Correspondence Principles for Effective Dimensions*

John M. Hitchcock
Department of Computer Science
Iowa State University
Ames, IA 50011
jhitchco@cs.iastate.edu

Abstract

We show that the classical Hausdorff and constructive dimensions of any union of Π^0_1-definable sets of binary sequences are equal. If the union is effective, that is, the set of sequences is Σ^0_2-definable, then the computable dimension also equals the Hausdorff dimension. This second result is implicit in the work of Staiger (1998).

Staiger also proved related results using entropy rates of decidable languages. We show that Staiger's computable entropy rate provides an equivalent definition of computable dimension. We also prove that a constructive version of Staiger's entropy rate coincides with constructive dimension.

1 Introduction

Lutz has recently effectivized classical Hausdorff dimension to define the constructive and computable dimensions of sets of infinite binary sequences [2, 3]. In early lectures on these effective dimensions [4], Lutz conjectured that there should be a *correspondence principle* stating that the constructive dimension of every sufficiently simple set X coincides with its classical Hausdorff dimension. In this paper we provide such a principle, along with an analogous correspondence principle for computable dimension. Specifically, given a set X of infinite binary sequences, let $\dim_{\mathrm{H}}(X)$ be the Hausdorff dimension of X, $\mathrm{cdim}(X)$ be the constructive dimension of X, and $\dim_{\mathrm{comp}}(X)$ be the computable dimension of X. Our correspondence principle for constructive dimension says that for every set X that is an *arbitrary* union of Π^0_1-definable sets of sequences, $\mathrm{cdim}(X) = \dim_{\mathrm{H}}(X)$. The correspondence principle for computable dimension says that for every Σ^0_2-definable set X of sequences, $\dim_{\mathrm{comp}}(X) = \dim_{\mathrm{H}}(X)$. We show that these results are optimal in the arithmetical hierarchy.

*This research was supported in part by National Science Foundation Grant 9988483.

Staiger [7] has proven closely related results in his investigations of Kolmogorov complexity and Hausdorff dimension. The correspondence principle for computable dimension is implicit in his results on martingale exponents of increase. In addition, for each set X of sequences he defined a kind of *entropy rate* that coincides with classical Hausdorff dimension. Staiger proved that a computable version of this entropy rate is equal to the classical Hausdorff dimension for Σ^0_2-definable sets. We show here that for every set X, Staiger's computable entropy rate of X coincides with the computable dimension of X. This provides a second proof of the correspondence principle for computable dimension. We also show that a constructive version of Staiger's entropy rate coincides with constructive dimension.

This paper is organized as follows. Section 2 contains the necessary preliminaries. In section 3 we review the Hausdorff, computable, and constructive dimensions. The correspondence principles are presented in section 4. The comparison of effective dimensions with effective entropy rates is given in section 5.

2 Preliminaries

We write $\{0,1\}^*$ for the set of all finite binary *strings* and $\mathbf{C}$ for the Cantor space of all infinite binary *sequences*. For $\omega \in \{0,1\}^* \cup \mathbf{C}$ and $i, j \in \mathbb{N}$, $\omega[i..j]$ is the string consisting of bits i through j of ω. For $w, v \in \{0,1\}^*$, we write $w \sqsubseteq v$ if w is a prefix of v. A *prefix set* is a language $A \subseteq \{0,1\}^*$ such that no element of A is a prefix of any other element of A. The sets of strings of length n and of length less than n are $\{0,1\}^n$ and $\{0,1\}^{<n}$. For any $A \subseteq \{0,1\}^*$, $A_{=n} = A \cap \{0,1\}^n$ and $A_{<n} = A \cap \{0,1\}^{<n}$.

We write DEC for the class of *decidable* languages and CE for the class of *computably enumerable* languages.

We will define the first two levels of the arithmetical hierarchy of subsets of $\mathbf{C}$. For each $w \in \{0,1\}^*$, the basic open set $\mathbf{C}_w$ is the set of all sequences in $\mathbf{C}$ that begin with prefix w. We let $\mathbf{C}_\top = \emptyset$.

Definition. Let $X \subseteq \mathbf{C}$.

- $X \in \Sigma^0_1$ if there is a computable function $h : \mathbb{N} \to \{0,1\}^* \cup \{\top\}$ such that

$$X = \bigcup_{i=0}^{\infty} \mathbf{C}_{h(i)}.$$

- $X \in \Pi^0_1$ if $X^c \in \Sigma^0_1$.

- $X \in \Sigma^0_2$ if there is a computable function $h : \mathbb{N} \times \mathbb{N} \to \{0,1\}^* \cup \{\top\}$ such that

$$X = \bigcup_{i=0}^{\infty} \bigcap_{j=0}^{\infty} \mathbf{C}^c_{h(i,j)}.$$

- $X \in \Pi^0_2$ if $X^c \in \Sigma^0_2$.

Note that every $X \in \Sigma^0_1$ is open and every $X \in \Pi^0_1$ is closed in the standard (product) topology on $\mathbf{C}$. In fact, Σ^0_1 and Π^0_1 are the *computably open* and *computably closed* subsets of $\mathbf{C}$, respectively. Recall that $\mathbf{C}$ is *compact*. This implies that for any closed set $X \subseteq \mathbf{C}$ and any collection of strings $A \subseteq \{0,1\}^*$ such that $X \subseteq \bigcup_{w \in A} \mathbf{C}_w$, there is a finite subcollection $A' \subseteq A$ such that $X \subseteq \bigcup_{w \in A'} \mathbf{C}_w$.

We say that a real-valued function $f : \{0,1\}^* \to [0,\infty)$ is *computable* if there is a computable function $\hat{f} : \mathbb{N} \times \{0,1\}^* \to [0,\infty) \cap \mathbb{Q}$ such that for all $n \in \mathbb{N}$ and $w \in \{0,1\}^*$, $|f(w) - \hat{f}(n,w)| \leq 2^{-n}$. A rational-valued function $f : \{0,1\}^* \to [0,\infty) \cap \mathbb{Q}$ that is itself computable is called *exactly computable*. We say that $f : \{0,1\}^* \to [0,\infty)$ is *lower semi-computable* if there is a computable function $g : \mathbb{N} \times \{0,1\}^* \to [0,\infty) \cap \mathbb{Q}$ such that for any $w \in \{0,1\}^*$, $g(n,w) \leq g(n+1,w) < f(w)$ for all $n \in \mathbb{N}$ and $f(w) = \lim_{n\to\infty} g(n,w)$.

3 Hausdorff, Constructive, and Computable Dimensions

In this section we briefly review classical Hausdorff dimension, constructive dimension, and computable dimension.

For each $k \in \mathbb{N}$, let $\mathcal{A}_k$ be the set of all prefix sets $A \subseteq \{0,1\}^*$ such that $A_{<k} = \emptyset$. For each $X \subseteq \mathbf{C}$, $s \in [0,\infty)$, and $k \in \mathbb{N}$, we define

$$H^s_k(X) = \inf \left\{ \sum_{w \in A} 2^{-s|w|} \,\middle|\, A \in \mathcal{A}_k \text{ and } X \subseteq \bigcup_{w \in A} \mathbf{C}_w \right\}$$

and

$$H^s(X) = \lim_{k\to\infty} H^s_k(X).$$

Definition. The *Hausdorff dimension* of a set $X \subseteq \mathbf{C}$ is

$$\dim_{\mathrm{H}}(X) = \inf \{ s \in [0,\infty) \,|\, H^s(X) = 0 \}.$$

For more information on Hausdorff dimension we refer to the book by Falconer [1].

Lutz [2] proved an alternative characterization of Hausdorff dimension using functions called gales and supergales. Gales and supergales are generalizations of martingales and supermartingales.

Definition. Let $s \in [0,\infty)$. A function $d : \{0,1\}^* \to [0,\infty)$ is an *s-supergale* if for all $w \in \{0,1\}^*$,

$$d(w) \geq \frac{d(w0) + d(w1)}{2^s}. \tag{3.1}$$

If equality holds in (3.1) for all $w \in \{0,1\}^*$, then d is an *s-gale*. A martingale is a 1-gale and a supermartingale is a 1-supergale.

Intuitively, a supergale is viewed as a function betting on an unknown binary sequence. If w is a prefix of the sequence, then the capital of the supergale after placing its first $|w|$ bets is given by $d(w)$. Assuming that w is a prefix of the sequence, the supergale places bets on $w0$ and $w1$ also being prefixes. The parameter s determines the fairness of the betting; as s decreases less capital is returned to the bettor. The goal of a supergale is to bet successfully on sequences.

Definition. Let $s \in [0, \infty)$ and let d be an s-supergale.

1. We say d *succeeds on* a sequence $S \in \mathbf{C}$ if
$$\limsup_{n\to\infty} d(S[0..n-1]) = \infty.$$

2. The *success set* of d is
$$S^\infty[d] = \{S \in \mathbf{C} \mid d \text{ succeeds on } S\}.$$

Theorem 3.1. (Lutz [2]) *For any $X \subseteq \mathbf{C}$,*
$$\dim_{\mathrm{H}}(X) = \inf \left\{ s \;\middle|\; \begin{array}{l} \textit{there exists a s-gale } d \\ \textit{for which } X \subseteq S^\infty[d] \end{array} \right\}.$$

This characterization of Hausdorff dimension motivates the following definitions of computable dimension [2] and constructive dimension [3]. We say that an s-supergale d is *constructive* if it is lower-semicomputable.

Definition. Let $X \subseteq \mathbf{C}$.

1. The *computable dimension* of X is
$$\dim_{\mathrm{comp}}(X) = \inf \left\{ s \;\middle|\; \begin{array}{l} \text{there exists a computable} \\ s\text{-gale } d \text{ for which } X \subseteq S^\infty[d] \end{array} \right\}.$$

2. The *constructive dimension* of X is
$$\mathrm{cdim}(X) = \inf \left\{ s \;\middle|\; \begin{array}{l} \text{there exists a constructive} \\ s\text{-supergale } d \text{ for which } X \subseteq S^\infty[d] \end{array} \right\}.$$

3. The *dimension* of a sequence $S \in \mathbf{C}$ is $\dim(S) = \mathrm{cdim}(\{S\})$.

Observe that for any set $X \subseteq \mathbf{C}$,
$$0 \le \dim_{\mathrm{H}}(X) \le \mathrm{cdim}(X) \le \dim_{\mathrm{comp}}(X) \le 1.$$

An important property of constructive dimension is the following *pointwise stability* property.

Lemma 3.2. (Lutz [3]) *For all $X \subseteq \mathbf{C}$,*

$$\mathrm{cdim}(X) = \sup_{S \in X} \dim(S).$$

The following exact computation lemma shows that computable dimension can be equivalently defined using exactly computable gales in place of computable gales.

Lemma 3.3. (Lutz [2]) *For any computable s-supergale d where 2^s is rational, there is an exactly computable s-gale d' such that $S^\infty[d] \subseteq S^\infty[d']$.*

Corollary 3.4. *For all $X \subseteq \mathbf{C}$, if $s > \dim_{\mathrm{comp}}(X)$ and 2^s is rational, then there is an exactly computable s-gale d such that $X \subseteq S^\infty[d]$.*

Proof. Let $X \subseteq \mathbf{C}$ and assume the hypothesis. Then for some $r < s$ there is a computable r-gale d_0 with $X \subseteq S^\infty[d_0]$. Since d_0 is an s-supergale, it follows from Lemma 3.3 that there is an exactly computable s-gale d with $X \subseteq S^\infty[d_0] \subseteq S^\infty[d]$. □

4 Correspondence Principles

In this section we will prove that $\mathrm{cdim}(X) = \dim_{\mathrm{H}}(X)$ for any X that is an *arbitrary* union of Π^0_1-definable sets. We will also show that $\dim_{\mathrm{comp}}(X) = \dim_{\mathrm{H}}(X)$ if X is Σ^0_2-definable.

Lemma 4.1. *If $X \in \Pi^0_1$, then $\dim_{\mathrm{H}}(X) = \dim_{\mathrm{comp}}(X)$.*

Proof. Let $X \in \Pi^0_1$. Since $\dim_{\mathrm{comp}}(X) \geq \dim_{\mathrm{H}}(X)$, it is enough to prove that $\dim_{\mathrm{comp}}(X) \leq \dim_{\mathrm{H}}(X)$. For this, let $t > \dim_{\mathrm{H}}(X)$ be arbitrary; it suffices to show that $\dim_{\mathrm{comp}}(X) \leq t$. Choose an s so that $\dim_{\mathrm{H}}(X) < s < t$ and 2^s is rational.

Since $s > \dim_{\mathrm{H}}(X)$, for each $r \in \mathbb{N}$, there is a prefix set $A_r \subseteq \{0,1\}^*$ such that

$$\sum_{w \in A_r} 2^{-s|w|} \leq 2^{-r} \text{ and } X \subseteq \bigcup_{w \in A_r} \mathbf{C}_w.$$

Because $\mathbf{C}$ is compact and X is closed, X is compact. Therefore each A_r may be taken finite.

Because $X \in \Pi^0_1$, there is a computable function $h : \mathbb{N} \to \{0,1\}^* \cup \{\top\}$ such that

$$X = \bigcap_{i=0}^{\infty} \mathbf{C}^c_{h(i)}.$$

For each $k \in \mathbb{N}$, let

$$X_k = \bigcap_{i=0}^{k} \mathbf{C}^c_{h(i)}.$$

Then for each $k \in \mathbb{N}$, it is easy to compute a finite prefix set B_k such that

$$\sum_{w \in B_k} 2^{-s|w|} \text{ is minimal and } X_k \subseteq \bigcup_{w \in B_k} \mathbf{C}_w.$$

For each $r \in \mathbb{N}$, let

$$k_r = \min\left\{ k \,\middle|\, \sum_{w \in B_k} 2^{-s|w|} \le 2^{-r} \right\}.$$

We know that each k_r exists because of the existence of the finite prefix sets A_r that satisfy the condition. Also, each k_r can be computed by computing the finite sets B_k until the condition is satisfied.

The rest of the proof is based on a construction used in characterizing Hausdorff dimension in terms of gales [2]. There the prefix sets A_r mentioned above are used to give an s-gale that succeeds on X. Here we use the finite, computable prefix sets B_{k_r} in the same manner to give a computable s-gale that succeeds on X.

Define for each $r \in \mathbb{N}$ a function $d_r : \{0,1\}^* \to [0,\infty)$ by

$$d_r(w) = \begin{cases} 2^{(s-1)(|w|-|v|)} & \text{if } (\exists v \sqsubseteq w) v \in B_{k_r} \\ \sum\limits_{\substack{u \in \{0,1\}^* \\ wu \in B_{k_r}}} 2^{-s|u|} & \text{otherwise.} \end{cases}$$

Notice that $d_r(\lambda) \le 2^{-r}$ and $d_r(w) = 1$ for all $w \in B_{k_r}$. For any string w, if w has a prefix $v \in B_{k_r}$ (which must be unique), then

$$\begin{aligned} d_r(w0) + d_r(w1) &= 2^{(s-1)(|w0|-|v|)} + 2^{(s-1)(|w1|-|v|)} \\ &= 2^s \cdot 2^{(s-1)(|w|-|v|)} \\ &= 2^s d_r(w). \end{aligned}$$

Otherwise,

$$\begin{aligned} d_r(w0) + d_r(w1) &= \sum_{\substack{u \in \{0,1\}^* \\ w0u \in B_{k_r}}} 2^{-s|u|} + \sum_{\substack{u \in \{0,1\}^* \\ w1u \in B_{k_r}}} 2^{-s|u|} \\ &= \sum_{\substack{b \in \{0,1\} \\ u \in \{0,1\}^* \\ wbu \in B_{k_r}}} 2^{-s|u|} \\ &= 2^s \sum_{\substack{u \in \{0,1\}^* \\ wu \in B_{k_r}}} 2^{-s|u|} \\ &= 2^s d_r(w), \end{aligned}$$

with the first equality holding even if wb has a prefix in B_{k_r} for $b = 0$ or $b = 1$ because then $wb \in B_{k_r}$ and

$$d_r(wb) = 1 = \sum_{\substack{u \in \{0,1\}^* \\ wbu \in B_{k_r}}} 2^{-s|u|}.$$

Therefore each d_r is an s-gale. Next define a function d on $\{0,1\}^*$ by $d = \sum_{r=0}^{\infty} 2^r d_{2r}$. Then

$$d(\lambda) = \sum_{r=0}^{\infty} 2^r d_{2r}(\lambda) \leq \sum_{r=0}^{\infty} 2^r 2^{-2r} = 2,$$

so by induction it follows that $d(w) < \infty$ for all strings w. Therefore, by linearity, d is an s-gale.

Let $S \in X$. For all $r \in \mathbb{N}$, we have $S \in X_{k_{2r}}$, so S has some prefix $S[0..n_r - 1] \in B_{k_{2r}}$. Then

$$d(S[0..n_r - 1]) \geq 2^r d_{2r}(S[0..n_r - 1]) = 2^r$$

for all $r \in \mathbb{N}$. Therefore d succeeds on S, so $X \subseteq S^{\infty}[d]$.

To see that d is computable, define $\hat{d} : \mathbb{N} \times \{0,1\}^* \to [0, \infty)$ by

$$\hat{d}(i, w) = \sum_{r=0}^{\lceil s|w| \rceil + i} 2^r d_{2r}(w).$$

We can exactly compute $\hat{d}$ by using the function h to uniformly compute the sets B_{k_r}. Then

$$\begin{aligned}
\left| d(w) - \hat{d}(i, w) \right| &= \sum_{r=\lceil s|w| \rceil + i + 1}^{\infty} 2^r d_{2r}(w) \\
&\leq \sum_{r=\lceil s|w| \rceil + i + 1}^{\infty} 2^r 2^{s|w|} d_{2r}(\lambda) \\
&\leq \sum_{r=\lceil s|w| \rceil + i + 1}^{\infty} 2^{r + s|w|} 2^{-2r} \\
&= 2^{s|w|} \sum_{r=\lceil s|w| \rceil + i + 1}^{\infty} 2^{-r} \\
&= 2^{s|w|} 2^{-\lceil s|w| \rceil - i} \\
&\leq 2^{-i},
\end{aligned}$$

so $\hat{d}$ is a computable approximation of d. Therefore d is computable, so it witnesses that $\dim_{\mathrm{comp}}(X) \leq s < t$. □

We now use the preceding lemma to give our correspondence principle for constructive dimension.

Theorem 4.2. *If $X \subseteq \mathbf{C}$ is a union of Π_1^0 sets, then* $\dim_{\mathrm{H}}(X) = \mathrm{cdim}(X)$.

Proof. Let $\mathcal{I}$ be an arbitrary index set, $X_\alpha \in \Pi_1^0$ for each $\alpha \in \mathcal{I}$, and $X = \bigcup_{\alpha \in \mathcal{I}} X_\alpha$. By definition, $\dim_{\mathrm{H}}(X) \leq \mathrm{cdim}(X)$. Using Lemma 3.2 (the pointwise stability of constructive dimension), Lemma 4.1, and the monotonicity of Hausdorff dimension, we have

$$\begin{aligned}\mathrm{cdim}(X) &= \sup_{\alpha \in \mathcal{I}} \mathrm{cdim}(X_\alpha) \\ &= \sup_{\alpha \in \mathcal{I}} \dim_{\mathrm{H}}(X_\alpha) \\ &\leq \dim_{\mathrm{H}}(X).\end{aligned}$$

□

Theorem 4.2 yields a pointwise characterization of the classical Hausdorff dimension of unions of Π^0_1 sets.

Corollary 4.3. *If $X \subseteq \mathbf{C}$ is a union of Π^0_1 sets, then*

$$\dim_{\mathrm{H}}(X) = \sup_{S \in X} \dim(S).$$

Proof. This follows immediately from Theorem 4.2 and Lemma 3.2. □

If we require that the union in Theorem 4.2 be effective, we arrive at the following correspondence principle for computable dimension. This result also follows implicitly from Staiger's work on martingale exponents of increase [7].

Theorem 4.4. *If $X \in \Sigma^0_2$, then $\dim_{\mathrm{H}}(X) = \dim_{\mathrm{comp}}(X)$.*

Proof. Let $X \in \Sigma^0_2$. Since $\dim_{\mathrm{comp}}(X) \geq \dim_{\mathrm{H}}(X)$, it is enough to prove that $\dim_{\mathrm{comp}}(X) \leq \dim_{\mathrm{H}}(X)$. For this, let $s > \dim_{\mathrm{H}}(X)$ be such that 2^s is rational. As in the proof of Lemma 4.1, it suffices to give a computable s-gale d that succeeds on X.

Since $X \in \Sigma^0_2$, there is a computable function $h : \mathbb{N} \times \mathbb{N} \to \{0,1\}^* \cup \{\top\}$ such that

$$X = \bigcup_{j=0}^{\infty} \bigcap_{i=0}^{\infty} \mathbf{C}^c_{h(i,j)}.$$

For each $j \in \mathbb{N}$, let

$$X_j = \bigcap_{i=0}^{\infty} \mathbf{C}^c_{h(i,j)}.$$

Since each $X_j \subseteq X$, $\dim_{\mathrm{H}}(X_j) \leq \dim_{\mathrm{H}}(X) < s$. Each $X_j \in \Pi^0_1$, so from the proof of Lemma 4.1, for each $j \in \mathbb{N}$, there is a computable s-gale d_j with $d_j(\lambda) \leq 2$ that succeeds on X_j. Let $d = \sum_{j=0}^{\infty} 2^{-j} d_j$. Then d is an s-gale, d is computable by using h to uniformly compute the d_j, and $X \subseteq S^\infty[d]$. □

We note that Theorems 4.2 and 4.4 cannot be extended to higher levels of the arithmetical hierarchy.

Observation 4.5. *There is a set $X \in \Pi^0_2$ such that $\dim_{\mathrm{H}}(X) \neq \mathrm{cdim}(X)$.*

Proof. It is well known that there exists a Martin-Löf random sequence $S \in \Delta^0_2$. (A sequence S is in Δ^0_2 if S is decidable relative to an oracle for the halting problem.) Let $X = \{S\}$. Since $S \in \Delta^0_2$, we have $X \in \Pi^0_2$. Lutz [3] observed that all random sequences have dimension 1, so $\mathrm{cdim}(X) = 1$. But any singleton has Hausdorff dimension 0, so $\dim_{\mathrm{H}}(X) = 0$. □

5 Dimension and Entropy Rates

In this section we compare our correspondence principles to related work of Staiger [7] on entropy rates. This comparison yields a new characterization of constructive dimension.

Definition. Let $A \subseteq \{0,1\}^*$. The *entropy rate of* A is

$$H_A = \limsup_{n\to\infty} \frac{\log |A_{=n}|}{n}.$$

(Here the logarithm is base 2 and we use the convention that $\log 0 = 0$.) Staiger observed that this entropy rate has a useful alternate characterization.

Lemma 5.1. (Staiger [6]) *For any* $A \subseteq \{0,1\}^*$,

$$H_A = \inf \left\{ s \,\middle|\, \sum_{w\in A} 2^{-s|w|} < \infty \right\}.$$

Definition. Let $A \subseteq \{0,1\}^*$. The *δ-limit of* A is

$$A^\delta = \{S \in \mathbf{C} \mid (\exists^\infty n) S[0..n-1] \in A\}.$$

That is, A^δ is the class of all sequences that have infinitely many prefixes in A.

For any $X \subseteq \mathbf{C}$, define

$$\mathcal{H}(X) = \{H_A \mid A \subseteq \{0,1\}^* \text{ and } X \subseteq A^\delta\},$$
$$\mathcal{H}_{\mathrm{DEC}}(X) = \{H_A \mid A \in \mathrm{DEC} \text{ and } X \subseteq A^\delta\},$$

and

$$\mathcal{H}_{\mathrm{CE}}(X) = \{H_A \mid A \in \mathrm{CE} \text{ and } X \subseteq A^\delta\}.$$

We call the infima of $\mathcal{H}(X), \mathcal{H}_{\mathrm{DEC}}(X)$, and $\mathcal{H}_{\mathrm{CE}}(X)$ the *entropy rate* of X, the *computable entropy rate* of X, and the *constructive entropy rate* of X, respectively.

Classical Hausdorff dimension may be characterized in terms of entropy rates.

Theorem 5.2. *For any* $X \subseteq \mathbf{C}$, $\dim_{\mathrm{H}}(X) = \inf \mathcal{H}(X)$.

A proof of Theorem 5.2 can be found in [6]; it also follows from Theorem 32 of [5].

Staiger proved the following relationship between computable entropy rates and Hausdorff dimension.

Theorem 5.3. (Staiger [7]) *For any* $X \in \Sigma^0_2$, $\dim_{\mathrm{H}}(X) = \inf \mathcal{H}_{\mathrm{DEC}}(X)$.

Putting Theorems 4.4, 5.2, and 5.3 together, for any $X \in \Sigma^0_2$ we have

$$\begin{array}{ccccc} \dim_{\mathrm{H}}(X) & = & \mathrm{cdim}(X) & = & \dim_{\mathrm{comp}}(X) \\ \| & & \| & & \| \\ \inf \mathcal{H}(X) & = & \inf \mathcal{H}_{\mathrm{CE}}(X) & = & \inf \mathcal{H}_{\mathrm{DEC}}(X). \end{array}$$

We will extend this to show that $\mathrm{cdim}(X) = \inf \mathcal{H}_{\mathrm{CE}}(X)$ and $\dim_{\mathrm{comp}}(X) = \inf \mathcal{H}_{\mathrm{DEC}}(X)$ hold for *arbitrary* $X \subseteq \mathbf{C}$. Note that the latter together with Theorem 5.3 provides a second proof of Theorem 4.4.

First we show that the dimensions are lower bounds of the entropy rates.

Lemma 5.4. *For any $X \subseteq \mathbf{C}$,*

$$\mathrm{cdim}(X) \leq \inf \mathcal{H}_{\mathrm{CE}}(X)$$

and

$$\dim_{\mathrm{comp}}(X) \leq \inf \mathcal{H}_{\mathrm{DEC}}(X).$$

Proof. We begin with a general construction that will be used to prove both inequalities. Let $A \subseteq \{0,1\}^*$ and let $t > s > H_A$. For each $n \in \mathbb{N}$, define a function $d_n : \{0,1\}^* \to [0,\infty)$ by

$$d_n(w) = \begin{cases} 2^{-t(n-|w|)} \cdot \left|\{v \in A_{=n} | w \sqsubseteq v\}\right| & \text{if } |w| \leq n \\ 2^{(t-1)(|w|-n)} d_n(w[0..n-1]) & \text{if } |w| > n. \end{cases}$$

Then each d_n is a t-gale. Define a function d on $\{0,1\}^*$ by $d = \sum_{n=0}^{\infty} 2^{(t-s)n} d_n$. Then

$$d(\lambda) = \sum_{n=0}^{\infty} 2^{(t-s)n} 2^{-tn} |A_{=n}| = \sum_{w \in A} 2^{-s|w|} < \infty$$

because $s > H_A$. By induction, $d(w) < \infty$ for all strings w, so $d : \{0,1\}^* \to [0,\infty)$. By linearity, d is also a t-gale. For any $w \in A$, we have

$$d(w) \geq 2^{(t-s)|w|} d_{|w|}(w) = 2^{(t-s)|w|},$$

so it follows that $A^\delta \subseteq S^\infty[d]$.

Let $r > \inf \mathcal{H}_{\mathrm{CE}}(X)$ be arbitrary. Then there is a computably enumerable A with $X \subseteq A^\delta$ and $H_A < r$. We can also choose 2^t and 2^s rational so that $H_A < s < t < r$. Because A is computably enumerable, the t-gale d defined above is constructive. Since $X \subseteq A^\delta \subseteq S^\infty[d]$, we have $\mathrm{cdim}(X) \leq t < r$. As this holds for all $r > \inf \mathcal{H}_{\mathrm{CE}}(X)$, we have $\mathrm{cdim}(X) \leq \inf \mathcal{H}_{\mathrm{CE}}(X)$.

If $\inf \mathcal{H}_{\mathrm{DEC}}(X) = 1$, then the inequality $\dim_{\mathrm{comp}}(X) \leq \inf \mathcal{H}_{\mathrm{DEC}}(X)$ is trivial, so assume $\inf \mathcal{H}_{\mathrm{DEC}}(X) < 1$. Let $1 > r > \inf \mathcal{H}_{\mathrm{DEC}}(X)$ be arbitrary. Take a decidable A and 2^s, 2^t rational such that $X \subseteq A^\delta$ and $H_A < s < t < r$. We will show that the t-gale d defined above is computable. For this, choose a natural number $k > \frac{1}{t-s}$. Define a function $\hat{d} : \{0,1\}^* \times \mathbb{N} \to [0,\infty) \cap \mathbb{Q}$ by

$$\hat{d}(w,r) = \sum_{n=0}^{kr+|w|} 2^{(s-t)n} d_n(w).$$

Then $\hat{d}$ is exactly computable. For all n, $d_n(w) \leq 1$ for all w with $|w| \geq n$, so for any precision r,

$$
\begin{aligned}
|d(w) - \hat{d}(w,r)| &= \sum_{n=kr+|w|+1}^{\infty} 2^{(s-t)n} d_n(w) \\
&\leq \sum_{n=kr+|w|+1}^{\infty} 2^{(s-t)n} \\
&\leq \sum_{n=kr+1}^{\infty} 2^{(s-t)n} \\
&= 2^{(s-t)(kr)} \\
&< 2^{-r}.
\end{aligned}
$$

Therefore $\hat{d}$ demonstrates that d is computable. Then $\dim_{\text{comp}}(X) \leq t < r$ because $X \subseteq A^{\delta} \subseteq S^{\infty}[d]$. It follows that $\dim_{\text{comp}}(X) \leq \inf \mathcal{H}_{\text{DEC}}(X)$ because $r > \inf \mathcal{H}_{\text{DEC}}(X)$ is arbitrary. □

Next we give lower bounds for constructive dimension and computable dimension by entropy rates.

Lemma 5.5. *For all $X \subseteq \mathbf{C}$,*

$$\inf \mathcal{H}_{\text{CE}}(X) \leq \text{cdim}(X)$$

and

$$\inf \mathcal{H}_{\text{DEC}}(X) \leq \dim_{\text{comp}}(X).$$

Proof. Suppose that d is an s-supergale with $X \subseteq S^{\infty}[d]$. Assume without loss of generality that $d(\lambda) < 1$ and let $A = \{w \mid d(w) > 1\}$. Then for all $n \in \mathbb{N}$,

$$\sum_{w \in \{0,1\}^n} d(w) \leq 2^{sn}$$

and $|A_{=n}| \leq 2^{sn}$. Also, $X \subseteq S^{\infty}[d] \subseteq A^{\delta}$. For any $t > s$,

$$\sum_{w \in A} 2^{-t|w|} = \sum_{n=0}^{\infty} 2^{-tn} |A_{=n}| \leq \sum_{n=0}^{\infty} 2^{(s-t)n} < \infty,$$

so $H_A \leq t$. Therefore $H_A \leq s$.

Let $s > \text{cdim}(X)$ such that there is a constructive s-supergale d succeeding on X. Then the set A defined above is computably enumerable, so $H_A \in \mathcal{H}_{\text{CE}}(X)$. We showed that $H_A \leq s$, so $\inf \mathcal{H}_{\text{CE}}(X) \leq s$. Therefore $\inf \mathcal{H}_{\text{CE}}(X) \leq \text{cdim}(X)$.

If $s > \dim_{\text{comp}}(X)$ and 2^s is rational, then by Corollary 3.4 there is an exactly computable s-gale d succeeding on X. Then the set A above is decidable, and analogously we obtain $\inf \mathcal{H}_{\text{DEC}}(X) \leq \dim_{\text{comp}}(X)$. □

Combining Lemmas 5.4 and 5.5 yields new characterizations of constructive and computable dimension.

Theorem 5.6. *For all* $X \subseteq \mathbf{C}$,

$$\mathrm{cdim}(X) = \inf \mathcal{H}_{\mathrm{CE}}(X)$$

and

$$\dim_{\mathrm{comp}}(X) = \inf \mathcal{H}_{\mathrm{DEC}}(X).$$

We remark that some resource-bounded analogues of these results hold. For example, if we define the similar concepts for polynomial-time and polynomial-space computability [2], the proofs of Lemmas 5.4 and 5.5 can be extended to show that

$$\dim_{\mathrm{p}}(X) \geq \inf \mathcal{H}_{\mathrm{P}}(X)$$

and

$$\dim_{\mathrm{pspace}}(X) = \inf \mathcal{H}_{\mathrm{PSPACE}}(X)$$

hold for all $X \subseteq \mathbf{C}$.

Acknowledgments. I thank Jack Lutz, Elvira Mayordomo, and the anonymous referees for helpful suggestions and remarks.

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