Dimension, Entropy Rates, and Compression

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Abstract

This paper develops new relationships between resource-bounded dimension, entropy rates, and compression. New tools for calculating dimensions are given and used to improve previous results about circuit-size complexity classes.

Approximate counting of SpanP functions is used to prove that the NP-entropy rate is an upper bound for dimension in Δ_3^{E}, the third level of the exponential-time hierarchy. This general result is applied to simultaneously improve the results of Mayordomo (1994) on the measure on P/poly in Δ_3^{E} and of Lutz (2000) on the dimension of exponential-size circuit complexity classes in ESPACE.

Entropy rates of efficiently rankable sets, sets that are optimally compressible, are studied in conjunction with time-bounded dimension. It is shown that rankable entropy rates give upper bounds for time-bounded dimensions. We use this to improve results of Lutz (1992) about polynomial-size circuit complexity classes from resource-bounded measure to dimension.

Exact characterizations of the effective dimensions in terms of Kolmogorov complexity rates at the polynomial-space and higher levels have been established, but in the time-bounded setting no such equivalence is known. We introduce the concept of polynomial-time superranking as an extension of ranking. We show that superranking provides an equivalent definition of polynomial-time dimension. From this superranking characterization we show that polynomial-time Kolmogorov complexity rates give a lower bound on polynomial-time dimension.

1 Introduction

Effective fractal dimension [29, 30] is an extension of Hausdorff dimension that provides new measures of complexity for classes of decision problems. The fractal dimension of a class can now be measured relative to a variety of levels of effectivization including finite-state, polynomial-time, polynomial-space, computable, and constructive bounds. These effective dimensions have interesting relationships with other measures of complexity including compressibility [30, 33, 7, 24], unpredictability [9, 15], and entropy rates [18]. Applications to circuit complexity [29, 19] and many other aspects of computational complexity have been given by several authors (see [1, 31, 20]).

For resource-bounds at polynomial-space and above, exact characterizations of the effective dimensions in terms of Kolmogorov complexity [30, 33, 14] and entropy rates [18], two different notions of compressibility, are known. For example, we have

$$\dim_{\mathrm{pspace}}(X) = \mathcal{H}_{\mathrm{PSPACE}}(X) = \mathcal{KS}^{\mathrm{poly}}(X)$$

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for any class X, where $\dim_{\rm pspace}$ is the polynomial-space dimension, $\mathcal{H}_{\rm PSPACE}$ is the PSPACE-entropy rate, and $\mathcal{KS}^{\rm poly}$ is a quantity defined using polynomial-space-bounded Kolmogorov complexity. (Definitions are given in the body of the paper.) At the polynomial-time level the equivalence proofs break down because it is not possible to perform an exponential search. This leaves us with $\dim_{\rm p}$, $\mathcal{H}_{\rm P}$, and $\mathcal{K}^{\rm poly}$ – dimension, entropy, and compression – as three possibly different measures of complexity at the polynomial-time level. We study these quantities and several related notions. Our results yield improvements of prior results about the resource-bounded measure and dimension of circuit-size complexity classes.

We find the NP-entropy rate $\mathcal{H}_{\rm NP}$ to be particularly useful for Boolean circuit-size complexity classes. For any X, we have

$$\dim_{\rm pspace}(X) \leq \mathcal{H}_{\rm NP}(X) \leq \mathcal{H}_{\rm P}(X) \leq \dim_{\rm p}(X).$$

Let $\mathrm{SIZE}(s(n))$ be the class of all languages that can be decided by nonuniform families of Boolean circuits of size at most $s(n)$. Lutz [29] used a polynomial-space counting technique to show that

$$\dim\left(\mathrm{SIZE}\left(\alpha\frac{2^n}{n}\right)\middle|\,\mathrm{ESPACE}\right) = \dim_{\rm pspace}\left(\mathrm{SIZE}\left(\alpha\frac{2^n}{n}\right)\right) = \alpha \tag{1.1}$$

for every $\alpha \in [0,1]$. Mayordomo [32] used Stockmeyer's approximate counting (in polynomial time with a $\Sigma_2^{\rm P}$-oracle) of #P functions to prove that P/poly has resource-bounded measure 0 in the third level of the exponential-time hierarchy:

$$\mu(\mathrm{P/poly} \mid \Delta_3^{\rm E}) = \mu_{\Delta_3^{\rm P}}(\mathrm{P/poly}) = 0. \tag{1.2}$$

In section 5 we strengthen (1.1) to $\Delta_3^{\rm P}$-dimension:

$$\dim\left(\mathrm{SIZE}\left(\alpha\frac{2^n}{n}\right)\middle|\,\Delta_3^{\rm E}\right) = \dim_{\Delta_3^{\rm P}}\left(\mathrm{SIZE}\left(\alpha\frac{2^n}{n}\right)\right) = \alpha. \tag{1.3}$$

As a corollary, (1.3) implies that

$$\dim(\mathrm{P/poly} \mid \Delta_3^{\rm E}) = \dim_{\Delta_3^{\rm P}}(\mathrm{P/poly}) = 0,$$

improving (1.2). Our proof of (1.3) comes in two steps. First we show that the NP-entropy rate is also α:

$$\mathcal{H}_{\rm NP}\left(\mathrm{SIZE}\left(\alpha\frac{2^n}{n}\right)\right) = \alpha.$$

We then use approximate counting of SpanP functions by Köbler, Schöning, and Toran [22] to prove a general theorem that

$$\dim_{\Delta_3^{\rm P}}(X) \leq \mathcal{H}_{\rm NP}(X)$$

for any class X. The use of SpanP functions rather than #P functions is crucial in our proof. We are able to get a much stronger result than (1.2) because using a SpanP function yields much greater precision. Before taking the approximation we are able to get an exact count and avoid a large amount of overcounting that happens with the #P function in Mayordomo's proof.

Köbler and Lindner [21] considered the measure of P/poly in the second level of the exponential hierarchy. They used pseudorandom generators and results of [27, 5] to show that

$$\mu_{\rm p}(\mathrm{NP}) \neq 0 \Rightarrow \mu(\mathrm{P/poly} \mid \Delta_2^{\rm EXP}) = 0. \tag{1.4}$$

In section 6, we use recent work of Shaltiel and Umans [36] on derandomization for approximate counting to improve (1.4) to a dimension result:

$$\mu_{\mathrm{p}}(\mathrm{NP}) \neq 0 \Rightarrow \left[\dim(\mathrm{P/poly} \mid \Delta_2^{\mathrm{E}}) = \dim_{\Delta_2^{\mathrm{P}}}(\mathrm{P/poly}) = 0\right].$$

We also establish an analogous conditional improvement of (1.3).

Two other results in resource-bounded measure besides (1.2) and (1.4) regarding P/poly were proved by Lutz [26]. He showed that

$$\mu(\mathrm{SIZE}(n^k) \mid \mathrm{EXP}) = \mu_{\mathrm{p}_2}(\mathrm{SIZE}(n^k)) = 0$$

for every $k \in \mathbb{N}$ and that $\mu_{\mathrm{p}_3}(\mathrm{P/poly}) = 0$. In section 7, we improve these results from measure 0 to dimension 0. Our proof of this uses general tools that we develop involving rankable [10] and printable [13, 4] sets. For example, we show that the p-rankable-entropy rate is an upper bound on p-dimension: for any X,

$$\dim_{\mathrm{p}}(X) \leq \mathcal{H}_{\text{p-rankable}}(X). \tag{1.5}$$

Following a preliminary version of this paper, Gu [11] considered the dimensions of some infinitely-often circuit-complexity classes. We use (1.5) to further examine infinitely-often classes in section 8.

In section 9 we investigate p-dimension and polynomial-time Kolmogorov complexity. As mentioned above, at higher levels of complexity exact characterizations of the resource-bounded dimensions in terms of Kolmogorov complexity have been established, but in the time-bounded setting no such equivalence is known. We introduce the concept of polynomial-time superranking and use it to give an equivalent definition of polynomial-time dimension. From this we show that $\mathcal{K}^{\mathrm{poly}}(X) \leq \dim_{\mathrm{p}}(X)$ for any X.

After the preliminaries in section 2, we review resource-bounded measure and dimension in section 3 and entropy rates in section 4. Sections 5-9 contain our results and section 10 concludes with a brief summary.

2 Preliminaries

The set of all finite binary strings is $\{0,1\}^*$. The empty string is denoted by λ. We use the standard enumeration of binary strings $s_0 = \lambda, s_1 = 0, s_2 = 1, s_3 = 00, \ldots$. For two strings $x, y \in \{0,1\}^*$, we say $x \leq y$ if x precedes y in the standard enumeration and $x < y$ if x precedes y and is not equal to y. We write $x - 1$ for the predecessor of x in the standard enumeration. We use the notation $x \sqsubseteq y$ to say that x is a prefix of y. The length of a string $x \in \{0,1\}^*$ is denoted by $|x|$.

All *languages* (decision problems) in this paper are encoded as subsets of $\{0,1\}^*$. For a language $A \subseteq \{0,1\}^*$, we define $A_{\leq n} = A \cap \{0,1\}^{\leq n}$ and $A_{=n} = A \cap \{0,1\}^n$.

The *Cantor space* of all infinite binary sequences is $\mathbf{C}$. We routinely identify a language $A \subseteq \{0,1\}^*$ with the element of Cantor space that is A's characteristic sequence according to the standard enumeration of binary strings. In this way each complexity class is identified with a subset of Cantor space. We write $A \upharpoonright n$ for the n-bit prefix of the characteristic sequence of A, and $A[n]$ for the n^{th}-bit of its characteristic sequence.

We use log for the base 2 logarithm.

Our definitions of most complexity classes are standard. We use DEC for the class of decidable languages and CE for the class of computably enumerable languages. For any function $s : \mathbb{N} \to \mathbb{N}$, SIZE$(s(n))$ is the class of all languages A where for all sufficiently large n, $A_{=n}$ can be decided by a circuit with no more than $s(n)$ gates.

As in [26, 29], we use Δ to represent a class of functions computable within a *resource bound*. The Δ used in this paper are

$$\begin{aligned}
\text{all} &= \{f \mid f : \{0,1\}^* \to \{0,1\}^*\} \\
\text{comp} &= \{f \mid f \text{ is computable}\} \\
\text{pspace} &= \{f \mid f \text{ is computable in } n^{O(1)} \text{ space}\} \\
\text{p} = \text{p}_1 &= \{f \mid f \text{ is computable in } n^{O(1)} \text{ time}\} \\
\text{p}_2 &= \{f \mid f \text{ is computable in } 2^{(\log n)^{O(1)}} \text{ time}\} \\
\text{p}_3 &= \{f \mid f \text{ is computable in } 2^{2^{(\log\log n)^{O(1)}}} \text{ time}\}
\end{aligned}$$

and for $k \geq 2$ the relativized class $\Delta_k^{\text{p}} = \text{p}^{\Sigma_{k-1}^{\text{P}}}$. We also define the complexity classes $\text{P}_1 = \text{P}$, $\text{P}_2 = \text{DTIME}(2^{(\log n)^{O(1)}})$, and $\text{P}_3 = \text{DTIME}(2^{2^{(\log\log n)^{O(1)}}})$.

A real-valued function $f : \{0,1\}^* \to [0,\infty)$ is Δ-computable if there is a function $\hat{f} : \mathbb{N} \times \{0,1\}^* \to [0,\infty)$ such that $|\hat{f}(n,w) - f(w)| \leq 2^{-n}$ for all n and w and $\hat{f} \in \Delta$ (where n is encoded in unary). We say that f is *exactly Δ-computable* if $f : \{0,1\}^* \to \mathbb{Q}$ and $f \in \Delta$.

Associated with each resource bound Δ is a complexity class $R(\Delta)$. We refer to [26, 29] for the general definition that involves functions called *constructors*. For the Δ we use in this paper, $R(\Delta)$ is as follows.

$$\begin{aligned}
R(\text{all}) &= \mathbf{C} \\
R(\text{comp}) &= \text{DEC} \\
R(\text{pspace}) &= \text{ESPACE} = \text{DSPACE}(2^{\text{linear}}) \\
R(\text{p}) &= \text{E} = \text{DTIME}(2^{\text{linear}}) \\
R(\text{p}_2) &= \text{EXP} = \text{DTIME}(2^{\text{polynomial}}) \\
R(\text{p}_3) &= \text{E}_3 = \text{DTIME}(2^{\text{quasipolynomial}}) \\
R(\Delta_k^{\text{p}}) &= \Delta_k^{\text{E}}
\end{aligned}$$

Here for each $k \geq 1$, $\Delta_k^{\text{E}} = \text{E}^{\Sigma_{k-1}^{\text{P}}}$ is a class in the exponential-time hierarchy.

3 Resource-Bounded Measure and Dimension

In this section we review the basics of resource-bounded measure [26] and dimension [29]. More background is available in the survey papers [28, 34, 31].

Definition. 1. A *martingale* is a function $d : \{0,1\}^* \to [0,\infty)$ such that for all $w \in \{0,1\}^*$,

$$d(w) = \frac{d(w0) + d(w1)}{2}.$$

2. Let $s \in [0,\infty)$. An *s-gale* is a function $d : \{0,1\}^* \to [0,\infty)$ such that for all $w \in \{0,1\}^*$,

$$d(w) = \frac{d(w0) + d(w1)}{2^s}.$$

Note that a martingale is a 1-gale. The sequences on which martingales and gales attain unbounded value is a central concept in resource-bounded measure and dimension.

Definition. Let $d : \{0,1\}^* \to [0,\infty)$.

1. Let $S \in \mathbf{C}$. We say that d *succeeds on* S if

$$\limsup_{n\to\infty} d(S\upharpoonright n) = \infty.$$

2. The *success set* of d is $S^\infty[d] = \{S \in \mathbf{C} \mid d \text{ succeeds on } S\}$.

We can now define resource-bounded measure [26], resource-bounded dimension [29], and constructive dimension [30]. In the following definition Δ can be any of the resource bounds defined in section 2.

Definition. Let Δ be a resource bound and let $X \subseteq \mathbf{C}$.

1. X has Δ-*measure 0*, and we write $\mu_\Delta(X) = 0$, if there is a Δ-computable martingale d with $X \subseteq S^\infty[d]$.
2. X has *measure 0 in* $R(\Delta)$, and we write $\mu(X \mid R(\Delta)) = 0$, if $\mu_\Delta(X \cap R(\Delta)) = 0$.
3. The Δ-*dimension of* X is

$$\dim_\Delta(X) = \inf\left\{ s \;\middle|\; \begin{array}{l}\text{there is a } \Delta\text{-computable} \\ s\text{-gale } d \text{ with } X \subseteq S^\infty[d]\end{array} \right\}.$$

4. The *dimension of* X *in* $R(\Delta)$ is $\dim(X \mid R(\Delta)) = \dim_\Delta(X \cap R(\Delta))$.
5. The *constructive dimension of* X is

$$\mathrm{cdim}(X) = \inf\left\{ s \;\middle|\; \begin{array}{l}\text{there is a lower semicomputable} \\ s\text{-gale } d \text{ with } X \subseteq S^\infty[d]\end{array} \right\}.$$ [1]

For the case $\Delta = \text{all}$, μ_{all} is equivalent to Lebesgue measure [40] and $\dim_{\text{all}}$ is equivalent to Hausdorff dimension [29]. The following theorem states some of the key properties of resource-bounded measure and dimension.

Theorem 3.1. (Lutz [26, 29]) *Let Δ, Δ' be resource bounds and let $X \subseteq \mathbf{C}$.*

1. $\mu_\Delta(R(\Delta)) \neq 0$.
2. $\dim_\Delta(X) \in [0,1]$.
3. *If* $\dim_\Delta(X) < 1$*, then* $\mu_\Delta(X) = 0$.
4. *If* $\Delta \subseteq \Delta'$ *and* $\mu_\Delta(X) = 0$*, then* $\mu_{\Delta'}(X) = 0$.
5. *If* $\Delta \subseteq \Delta'$*, then* $\dim_{\Delta'}(X) \leq \dim_\Delta(X)$.

[1] The definition of constructive dimension given here is not the original one but was shown equivalent by Fenner [8] and Hitchcock [16].

Resource-bounded dimension admits an equivalent definition in terms of resource-bounded unpredictability in the log-loss model [15]. In [17], this characterization was restated in a useful way involving the log-loss of measures.

Definition. A *submeasure* is a function $\rho : \{0,1\}^* \to [0,\infty)$ such that for all $w \in \{0,1\}^*$,

$$\rho(w) \geq \rho(w0) + \rho(w1). \tag{3.1}$$

If equality holds in (3.1) for all $w \in \{0,1\}^*$, then ρ is a *measure*.

1. Let $S \in \mathbf{C}$. The *log-loss rate* of ρ on S is

$$\mathcal{L}^{\log}(\rho, S) = \liminf_{n\to\infty} \frac{-\log \rho(S \upharpoonright n)}{n}.$$

2. Let $X \subseteq \mathbf{C}$. The *worst case log-loss rate* of ρ on X is

$$\mathcal{L}^{\log}(\rho, X) = \sup_{S\in X} \mathcal{L}^{\log}(\rho, S).$$

Theorem 3.2. (Hitchcock [15, 17]) *Let Δ be a resource bound. For any $X \subseteq \mathbf{C}$,*

$$\dim_\Delta(X) = \inf\left\{ \mathcal{L}^{\log}(\rho, X) \middle| \rho \in \Delta \textit{ is a submeasure}\right\}.$$

Equality still holds when the infimum is taken over exactly Δ-computable measures ρ.

4 Entropy Rates

In this section we review entropy rates of languages and their relationship to dimension. The following concept dates back to Chomsky and Miller [6] and Kuich [23].

Definition. Let $A \subseteq \{0,1\}^*$. The *entropy rate of A* is

$$H_A = \limsup_{n\to\infty} \frac{\log |A_{=n}|}{n}.$$

Intuitively, H_A gives an asymptotic measurement of the amount by which every string in $A_{=n}$ is compressed in an optimal code.

Definition. Let $A \subseteq \{0,1\}^*$. The *i.o.-class of A* is

$$A^{\text{i.o.}} = \{S \in \mathbf{C} \mid (\exists^\infty n) S \upharpoonright n \in A\}.$$

That is, $A^{\text{i.o.}}$ is the class of sequences that have infinitely many prefixes in A. The name *δ-limit of A* and notation A^δ have also been used for $A^{\text{i.o.}}$ [37, 38].

Definition. Let $\mathcal{C}$ be a class of languages and $X \subseteq \mathbf{C}$. The *$\mathcal{C}$-entropy rate* of X is

$$\mathcal{H}_{\mathcal{C}}(X) = \inf\{H_A \mid A \in \mathcal{C} \text{ and } X \subseteq A^{\text{i.o.}}\}.$$

Informally, $\mathcal{H}_{\mathcal{C}}(X)$ is the lowest entropy rate with which every element of X can be covered infinitely often by a language in $\mathcal{C}$.

For all $X \subseteq \mathbf{C}$, classical results (see [35, 37]) imply

$$\dim_{\mathrm{H}}(X) = \mathcal{H}_{\mathrm{ALL}}(X),$$

where ALL is the class of all languages and $\dim_{\mathrm{H}}$ is Hausdorff dimension. Using other classes of languages gives equivalent definitions of the constructive, computable, and polynomial-space dimensions.

Theorem 4.1. (Hitchcock [18, 14]) *For all $X \subseteq \mathbf{C}$,*

$$\mathrm{cdim}(X) = \mathcal{H}_{\mathrm{CE}}(X),\ \dim_{\mathrm{comp}}(X) = \mathcal{H}_{\mathrm{DEC}}(X),\ \textit{and}\ \dim_{\mathrm{pspace}}(X) = \mathcal{H}_{\mathrm{PSPACE}}(X).$$

For time-bounded dimension no analogous result is known. However, the following upper bound is true.

Lemma 4.2. (Hitchcock [18, 14]) *For all $X \subseteq \mathbf{C}$,*

$$\mathcal{H}_{\mathrm{P}_i}(X) \leq \dim_{\mathrm{p}_i}(X).$$

Proof. Let $s > \dim_{\mathrm{p}_i}(X)$ such that 2^s is rational. It suffices to show $\mathcal{H}_{\mathrm{P}_i}(X) \leq s$. By Theorem 3.2 there is an exactly p_i-computable measure ρ with $\mathcal{L}^{\log}(\rho, S) < s$ for all $S \in X$. Define $A = \{w \mid \rho(w) \geq 2^{-s|w|}\}$. Then $A \in \mathrm{P}_i$ and $X \subseteq A^{\mathrm{i.o.}}$. Since ρ is a measure, $|A_{=n}| \leq 2^{sn}\rho(\lambda)$ for all n, so $H_A \leq s$. Therefore $\mathcal{H}_{\mathrm{P}_i}(X) \leq s$. □

We will consider $\mathcal{H}_{\mathcal{C}}$ for other complexity classes $\mathcal{C}$ including NP and the p-rankable sets. The following proposition shows that if $\mathcal{C}$ satisfies mild restrictions, then $\mathcal{H}_{\mathcal{C}}$ gives a reasonable notion of an effective dimension with many of the standard properties of the usual effective dimensions.

Proposition 4.3. *Let $\mathcal{C}, \mathcal{D}$ be classes of languages and $X, Y \subseteq \mathbf{C}$.*

1. *If $X \subseteq Y$, then $\mathcal{H}_{\mathcal{C}}(X) \leq \mathcal{H}_{\mathcal{C}}(Y)$.*

2. *If $\mathcal{C} \subseteq \mathcal{D}$, then $\mathcal{H}_{\mathcal{D}}(X) \leq \mathcal{H}_{\mathcal{C}}(X)$.*

3. *If $\{0,1\}^* \in \mathcal{C}$, then $\mathcal{H}_{\mathcal{C}}(\mathbf{C}) = 1$ and $0 \leq \mathcal{H}_{\mathcal{C}}(X) \leq 1$.*

4. *If $\mathcal{C}$ is closed under union, then $\mathcal{H}_{\mathcal{C}}(X \cup Y) = \max\{\mathcal{H}_{\mathcal{C}}(X), \mathcal{H}_{\mathcal{C}}(Y)\}$.*

5 Approximate Counting and Dimension in Δ_3^{E}

Mayordomo [32] used Stockmeyer's approximate counting of #P functions [39] to show that P/poly has measure 0 in the third level of the exponential hierarchy.

Theorem 5.1. (Mayordomo [32])

$$\mu(\mathrm{P/poly} \mid \Delta_3^{\mathrm{E}}) = \mu_{\Delta_3^{\mathrm{P}}}(\mathrm{P/poly}) = 0.$$

Lutz [29] calculated the dimension in ESPACE of some exponential circuit-size complexity classes.

Theorem 5.2. (Lutz [29]) *For all* $\alpha \in [0,1]$,

$$\dim\left(\mathrm{SIZE}\left(\alpha\frac{2^n}{n}\right)\middle|\mathrm{ESPACE}\right) = \dim_{\mathrm{pspace}}\left(\mathrm{SIZE}\left(\alpha\frac{2^n}{n}\right)\right) = \alpha.$$

In this section we will improve Theorems 5.1 and 5.2. We first show that the NP-entropy rate is also α for the classes in Theorem 5.2.

Theorem 5.3. *For all* $\alpha \in [0,1]$,

$$\mathcal{H}_{\mathrm{NP}}\left(\mathrm{SIZE}\left(\alpha\frac{2^n}{n}\right)\right) = \alpha.$$

Proof. Let $\alpha \in [0,1]$ and $s(n) = \alpha\frac{2^n}{n}$. Let

$$A = \{B_{\leq n} \mid (\forall m, \lfloor \log n \rfloor \leq m \leq n) B_{=m} \text{ has a circuit of size at most } s(m)\}.$$

Here we use $B_{\leq n}$ to denote the characteristic string (of length $2^{n+1}-1$) of a language B on strings up to length n. We have $A \in \mathrm{NP}$ and $\mathrm{SIZE}(s(n)) \subseteq A^{\mathrm{i.o.}}$.

Also, for all m, we know from [26] that there are at most $(48es(m))^{s(m)}$ *novel* m-input circuits of size at most $s(m)$. Here a circuit is novel if it does not compute the same function as any circuit of size at most $s(m)$ that precedes it in a lexicographic enumeration. This gives us an upper bound on how many subsets of $\{0,1\}^m$ have a circuit of size at most $s(m)$. We then have

$$\begin{aligned}
\log |A_{\leq 2^{n+1}-1}| &\leq \sum_{m=0}^{\lfloor \log n\rfloor -1} 2^m + \sum_{m=\lfloor \log n \rfloor}^{n} \log(48es(m))^{s(m)} \\
&\leq 2^{\log n} + \sum_{m=0}^{n} \log(48es(m))^{s(m)} \\
&= n + \sum_{m=0}^{n} \alpha\frac{2^m}{m}(m - \log m + \log 48e\alpha) \\
&\leq \alpha(2^{n+1}-1)
\end{aligned}$$

if n is sufficiently large, so $H_A \leq \alpha$. Therefore $\mathcal{H}_{\mathrm{NP}}(\mathrm{SIZE}(s(n))) \leq \alpha$.

The other inequality follows from Proposition 4.3(2) and Theorems 4.1 and 5.2. We have

$$\begin{aligned}
\mathcal{H}_{\mathrm{NP}}(\mathrm{SIZE}(s(n))) &\geq \mathcal{H}_{\mathrm{PSPACE}}(\mathrm{SIZE}(s(n))) \\
&= \dim_{\mathrm{pspace}}(\mathrm{SIZE}(s(n))) \\
&= \alpha.
\end{aligned}$$

□

We will make use of SpanP functions to prove a general theorem relating the $\mathcal{H}_{\mathrm{NP}}$ entropy rate to dimension in Δ_3^{E}. Köbler, Schöning, and Toran [22] introduced SpanP as an extension of #P.

Definition. Let M be a polynomial-time nondeterministic Turing machine that on each computation path either outputs a string or outputs nothing. The SpanP function computed by M is defined as

$$f(x) = \text{number of distinct strings output by } M \text{ on input } x$$

for all $x \in \{0,1\}^*$.

Every #P function is also a SpanP function. Stockmeyer's approximate counting of #P functions in polynomial-time with a Σ_2^{P} oracle extends to SpanP.

Theorem 5.4. (Köbler, Schöning, and Toran [22]) *Let $f \in \mathrm{SpanP}$. Then there is a function $g \in \Delta_3^{\mathrm{p}}$ such that for all n, for all $x \in \{0,1\}^n$,*

$$(1 - 1/n)g(x) \leq f(x) \leq (1 + 1/n)g(x).$$

We now show that the NP-entropy rate is an upper bound for Δ_3^{p}-dimension.

Theorem 5.5. *For all $X \subseteq \mathbf{C}$,*

$$\dim_{\Delta_3^{\mathrm{p}}}(X) \leq \mathcal{H}_{\mathrm{NP}}(X).$$

Proof. Let $\alpha > \mathcal{H}_{\mathrm{NP}}(X)$ and $\epsilon > 0$ such that 2^α, 2^ϵ are rational. Let $A \in \mathrm{NP}$ such that $X \subseteq A^{\mathrm{i.o.}}$ and $H_A < \alpha$. We can assume that $|A_{=n}| \leq 2^{\alpha n}$ for all n. It suffices to show that $\dim_{\Delta_3^{\mathrm{p}}}(X) \leq \alpha + \epsilon$.

For each n and $v \in \{0,1\}^{\leq n}$, let

$$\mathrm{ext}_A(v,n) = |\{v' \in A_{=n} \mid v \sqsubseteq v'\}|$$

be the number of extensions of v in $A_{=n}$. Define a function $f : 0^* \times \{0,1\}^* \to \mathbb{N}$ by

$$f(0^n, v) = \mathrm{ext}_A(v, n).$$

Then $f \in \mathrm{SpanP}$ by the following nondeterministic algorithm.

input $0^n, v$
guess $v' \in \{0,1\}^n$ with $v \sqsubseteq v'$
guess a witness w
if w witnesses that $v' \in A$
  then output v'
  else output nothing

Note that f has the following properties for all $n \in \mathbb{N}$.

- $f(0^n, \lambda) = |A_{=n}| \leq 2^{\alpha n}$.
- $f(0^n, v) = f(0^n, v0) + f(0^n, v1)$ for all $v \in \{0,1\}^{<n}$.
- $f(0^n, v) = 1$ for all $v \in A_{=n}$.

Let $g \in \Delta_3^{\mathrm{p}}$ be the approximation of f from Theorem 5.4. For each n, let $\epsilon_n = \frac{1}{n}$ and define a function ρ_n by

$$\rho_n(v) = \frac{g(0^n, v)}{2^{\alpha n}} \left(\frac{1 - \epsilon_n}{1 + \epsilon_n} \right)^{|v|}$$

for all $v \in \{0,1\}^{\leq n}$ and $\rho_n(v) = 2^{-(|v|-n)} \rho_n(v \upharpoonright n)$ for all v with $|v| > n$. Using the fact that

$$g(0^n, v0) + g(0^n, v1) \leq \frac{f(0^n, v0) + f(0^n, v1)}{1 - \epsilon_n} = \frac{f(0^n, v)}{1 - \epsilon_n} \leq g(0^n, v) \frac{1 + \epsilon_n}{1 - \epsilon_n},$$

we have

$$\begin{aligned}\rho_n(v0)+\rho_n(v1) &= \frac{g(0^n,v0)+g(0^n,v1)}{2^{\alpha n}}\left(\frac{1-\epsilon_n}{1+\epsilon_n}\right)^{|v|+1}\\ &\leq \frac{g(0^n,v)}{2^{\alpha n}}\left(\frac{1-\epsilon_n}{1+\epsilon_n}\right)^{|v|}\\ &= \rho_n(v),\end{aligned}$$

for all $v \in \{0,1\}^{<n}$, so ρ_n is a submeasure.

Let $v \in A_{=n}$. Then

$$-\log \rho_n(v) = \alpha n - \log g(0^n,v) + n \log \frac{1+\epsilon_n}{1-\epsilon_n},$$

$$g(0^n,v) \geq \frac{f(0^n,v)}{1+\epsilon_n} = \frac{1}{1+\epsilon_n} \geq \frac{1}{2},$$

and

$$\lim_{n\to\infty} n \log \frac{1+\epsilon_n}{1-\epsilon_n} = 2\log e,$$

so

$$-\log \rho_n(v) \leq \alpha n + 4$$

if n is sufficiently large.

Define $\rho = \sum_{n=0}^{\infty} 2^{-\epsilon n}\rho_n$. Standard techniques show that ρ is Δ_3^{p}-computable. Let $S \in X$. Then $S \in A^{\mathrm{i.o.}}$, so $S\restriction n \in A_{=n}$ for infinitely many n. Therefore

$$\liminf_{n\to\infty} \frac{-\log \rho_n(S\restriction n)}{n} \leq \alpha.$$

It follows that $\mathcal{L}^{\log}(\rho,S) \leq \alpha+\epsilon$ for all $S \in X$, so $\mathcal{L}^{\log}(\rho,X) \leq \alpha+\epsilon$. By Theorem 3.2 we have that the Δ_3^{p}-dimension of X is at most $\alpha+\epsilon$. □

We can now simultaneously improve Theorems 5.1 and 5.2.

Theorem 5.6. *For all* $\alpha \in [0,1]$,

$$\dim_{\Delta_3^{\mathrm{p}}}\left(\mathrm{SIZE}\left(\alpha\frac{2^n}{n}\right)\right) = \alpha.$$

Proof. The upper bound is immediate from Theorems 5.3 and 5.5. The lower bound follows from Theorems 5.2 and 4.1. □

Corollary 5.7. $\dim(\mathrm{P/poly} \mid \Delta_3^{\mathrm{E}}) = \dim_{\Delta_3^{\mathrm{p}}}(\mathrm{P/poly}) = 0.$

Next we show that the classes in Theorem 5.6 have dimension α in Δ_3^{E}. This proof is inspired by a technique of Gu [12].

Theorem 5.8. *For all* $\alpha \in (0,1)$,

$$\dim\left(\mathrm{SIZE}\left(\alpha\frac{2^n}{n}\right)\middle| \Delta_3^{\mathrm{E}}\right) = \alpha.$$

Proof. Let $s(n) = \alpha\frac{2^n}{n}$. We need to show that $\dim_{\Delta_3^{\mathrm{p}}}(\mathrm{SIZE}(s(n)) \cap \Delta_3^{\mathrm{E}}) \geq \alpha$. For this, let $s < t < \alpha$ be rational and let d be an arbitrary Δ_3^{p}-computable s-gale. Assume without loss of generality that $d(\lambda) = 1$ and d is exactly Δ_3^{p}-computable [29]. It suffices to show that $\mathrm{SIZE}(s(n)) \cap \Delta_3^{\mathrm{E}} \not\subseteq S^\infty[d]$.

We define a language A inductively as follows. Suppose that $A_{<n}$ has already been defined, and let w be the characteristic string of $A_{<n}$. As an inductive hypothesis assume that $d(w) \leq 1$. Define u of length $\lceil t2^n \rceil$ inductively by starting with $u = \lambda$ and repeatedly updating $u := u1$ if $d(wu1) < d(wu0)$, $u := u0$ if $d(wu1) \geq d(wu0)$. Let $v = 0^{2^n - |u|}$. Then for all $u' \sqsubseteq u$,

$$d(wu') \leq 2^{(s-1)|u'|} d(w) \leq 2^{(s-1)|u'|} \leq 1,$$

and for all $v' \sqsubseteq v$,

$$d(wuv') \leq 2^{s|v'|} d(wu) \leq 2^{s|uv'| - |u|} \leq 2^{s2^n - |u|} \leq 1.$$

We let $A_{=n}$ have characteristic string uv.

Since d never gets above 1 on A, we have $A \notin S^\infty[d]$, and by construction $A \in \Delta_3^{\mathrm{E}}$. We only sketch the argument that $A \in \mathrm{SIZE}(s(n))$. Let B_n be the first $\lceil t2^n \rceil$ strings of length n and let f be a mapping of B_n to $\{0,1\}^m$, where $m = \lceil \log \lceil t2^n \rceil \rceil$. Let A'_n be the image of $A_{=n} \cap B_n$ under f. For $\epsilon > 0$ and sufficiently large n, Lupanov's construction [25] yields a circuit L_n of size at most $\frac{2^m}{m}(1 + \epsilon)$ for A'_n. We now describe our circuit C_n for $A_{=n}$. First the circuit checks if the input x is in B_n. If $x \notin B_n$, C_n rejects. Otherwise, C_n computes $f(x)$ and applies L_n to $f(x)$. Since checking membership in B_n and computing f can both be done by polynomial-size circuits, C_n can be implemented in fewer than $s(n)$ gates if n is sufficiently large. □

6 Derandomization and Dimension in Δ_2^{E}

Köbler and Lindner used pseudorandom generators to prove that P/poly has measure 0 in the second level of the EXP-hierarchy if NP does not have p-measure 0.

Theorem 6.1. (Köbler and Lindner [21]) *If* $\mu_{\mathrm{p}}(\mathrm{NP}) \neq 0$, *then* $\mu(\mathrm{P/poly} \mid \mathrm{EXP}^{\mathrm{NP}}) = 0$.

We will improve this to dimension 0 in $\Delta_2^{\mathrm{E}} = \mathrm{E}^{\mathrm{NP}}$ ($\subseteq \mathrm{EXP}^{\mathrm{NP}}$) under the same hypothesis. For this we will use better approximate counting arising from derandomization. Recall that Stockmeyer [39] showed that #P functions can be approximated in randomized polynomial time with access to an NP oracle. This was extended to SpanP by Köbler, Schöning, and Toran [22].

Shaltiel and Umans [36] showed that under a derandomization assumption, #P functions can be approximated by a deterministic polynomial-time algorithm with nonadaptive access to an NP oracle. Their proof shows how to approximate the acceptance probability of a Boolean circuit. We observe that this proof also goes through for nondeterministic circuits, yielding the following. (For definitions of undefined concepts we refer to [36].)

Theorem 6.2. (Shaltiel and Umans [36]) *If* $\mathrm{E}_{\|}^{\mathrm{NP}}$ *requires exponential-size SV-nondeterministic circuits, then there is a deterministic algorithm that takes as inputs a nondeterministic circuit* C *and a parameter* $\epsilon > 0$, *runs in time polynomial in* $|C|$ *and* $1/\epsilon$ *making nonadaptive queries to an* NP *oracle, and outputs a real number* ρ *such that*

$$(1 - \epsilon)\mathrm{Pr}_x[C(x) = 1] \leq \rho \leq \mathrm{Pr}_x[C(x) = 1].$$

It follows that under the hypothesis of Theorem 6.2, SpanP functions can also be deterministically approximated with an NP oracle.

Corollary 6.3. *If* $\mathrm{E}_{\|}^{\mathrm{NP}}$ *requires exponential-size SV-nondeterministic circuits, then for any function* $f \in \mathrm{SpanP}$ *there is a function* g *computable in polynomial time with nonadaptive access to an* NP *oracle such that for all* n*, for all* $x \in \{0,1\}^n$*,*

$$g(x) \leq f(x) \leq g(x)(1 + 1/n).$$

Proof. Let $f \in \mathrm{SpanP}$ and let M be the nondeterministic polynomial-time machine defining f. We assume that on an input of length n, all outputs of M have length $p(n)$, where p is some polynomial. For any input x, define a nondeterministic circuit C_x that on an input $y \in \{0,1\}^{p(n)}$ simulates M and accepts if M outputs y. Applying Theorem 6.2 with $\epsilon = 1/(n+1)$, we can compute a number ρ_x that is a good approximation of the acceptance probability of C_x. Defining $g(x) = 2^{p(n)}\rho_x$, we have $(1 - \frac{1}{n+1})f(x) \leq g(x) \leq f(x)$, which implies the corollary. □

We can use this result to give a conditional improvement to Theorem 5.5.

Theorem 6.4. *If* $\mathrm{E}_{\|}^{\mathrm{NP}}$ *requires exponential-size SV-nondeterministic circuits, then*

$$\dim_{\Delta_2^{\mathrm{P}}}(X) \leq \mathcal{H}_{\mathrm{NP}}(X)$$

for all $X \subseteq \mathbf{C}$*.*

Proof. Use the approximation function from Corollary 6.3 in the proof of Theorem 5.5. □

The hypothesis of Theorem 6.4 can also be replaced by an assumption on the complexity of E^{NP} (revisiting the proof of Theorem 6.2), but the above suffices for our purposes. In particular, we have the following corollary.

Corollary 6.5. *If* $\mu_{\mathrm{p}}(\mathrm{NP}) \neq 0$*, then*

$$\dim_{\Delta_2^{\mathrm{P}}}(X) \leq \mathcal{H}_{\mathrm{NP}}(X)$$

for all $X \subseteq \mathbf{C}$*.*

Proof. It follows from the proof of Lemma 3.2 in [27] that if $\mu_{\mathrm{p}}(\mathrm{NP}) \neq 0$, then $\mathrm{NE} \subseteq \mathrm{E}_{\|}^{\mathrm{NP}}$ has exponential-size NP-oracle circuit complexity. □

We now have the following extension of Theorem 5.6.

Theorem 6.6. *If* $\mu_{\mathrm{p}}(\mathrm{NP}) \neq 0$*, then*

$$\dim_{\Delta_2^{\mathrm{P}}}\left(\mathrm{SIZE}\left(\alpha\frac{2^n}{n}\right)\right) = \alpha$$

for all $\alpha \in [0,1]$*.*

The improvement of Theorem 6.1 now follows.

Corollary 6.7. *If* $\mu_{\mathrm{p}}(\mathrm{NP}) \neq 0$*, then*

$$\dim_{\Delta_2^{\mathrm{P}}}(\mathrm{P/poly}) = \dim(\mathrm{P/poly} \mid \Delta_2^{\mathrm{E}}) = \dim(\mathrm{P/poly} \mid \Delta_2^{\mathrm{EXP}}) = 0.$$

7 Ranking, Printing, and Time-Bounded Dimension

Lutz [26] proved the following regarding the resource-bounded measure of polynomial-size circuit complexity classes.

Theorem 7.1. (Lutz [26]) *For all* $c \geq 1$,

$$\mu(\mathrm{SIZE}(n^c) \mid \mathrm{EXP}) = \mu_{\mathrm{p}_2}(\mathrm{SIZE}(n^c)) = 0$$

and

$$\mu(\mathrm{P/poly} \mid \mathrm{E}_3) = \mu_{\mathrm{p}_3}(\mathrm{P/poly}) = 0.$$

In this section we develop some tools involving rankable [10] and printable [13, 4] sets for calculating dimensions. These tools will yield a strengthening of Theorem 7.1 from measure 0 to dimension 0.

Definition. Let $A \subseteq \{0,1\}^*$.

1. A is p_i*-rankable* if the *ranking function* $\mathrm{rank}_A(x) = |\{y \in A \mid y \leq x\}|$ is in p_i.
2. A is p_i*-printable* if there is a function $f \in \mathrm{p}_i$ such that for all $n \in \mathbb{N}$, $f(0^n)$ lists all strings in $A_{=n}$.

While it is not known if $\dim_{\mathrm{p}_i}(X) \leq \mathcal{H}_{\mathrm{P}_i}(X)$ holds in general, we can show that the p_i-rankable-entropy rate is an upper bound on p_i-dimension.

Theorem 7.2. *For any* $X \subseteq \mathbf{C}$,

$$\dim_{\mathrm{p}_i}(X) \leq \mathcal{H}_{\mathrm{p}_i\text{-rankable}}(X).$$

Proof. We give the proof for $i = 1$; the other cases are entirely analogous. Let $t > s > \mathcal{H}_{\text{p-rankable}}(X)$ with $2^s \in \mathbb{Q}$. Choose $A \in$ p-rankable with $X \subseteq A^{\mathrm{i.o.}}$ and $H_A < s$. It suffices to show that $\dim_{\mathrm{p}}(X) \leq t$.

For each n and $w \in \{0,1\}^{\leq n}$, let

$$\mathrm{ext}_A(w,n) = |\{v \in A_{=n} \mid w \sqsubseteq v\}|$$

be the number of extensions of w in $A_{=n}$. Define

$$\rho_n(w) = \frac{\mathrm{ext}_A(w,n)}{2^{sn}}.$$

For w with $|w| > n$, we let $\rho_n(w) = 2^{-(|w|-n)}\rho_n(w\!\upharpoonright\! n)$. Note that for all $w \in \{0,1\}^{\leq n}$,

$$\mathrm{ext}_A(w,n) = \mathrm{rank}_A(w1^{n-|w|}) - \mathrm{rank}_A(w0^{n-|w|} - 1),$$

so $\mathrm{ext}_A(w,n)$ can be computed in time polynomial in n because A is p-rankable. Let $\epsilon \in (0, t-s)$ with $2^\epsilon \in \mathbb{Q}$ and define $\rho = \sum_{n=0}^{\infty} 2^{-\epsilon n}\rho_n$. Then ρ is a p-computable submeasure. Also, for any $w \in A$, we have $\rho_{|w|}(w) = 2^{-s|w|}$ and

$$\begin{aligned} -\log \rho(w) &\leq -\log 2^{-\epsilon|w|}\rho_{|w|}(w) \\ &= (s+\epsilon)|w| \\ &< t|w|. \end{aligned}$$

It follows from Theorem 3.2 that $\dim_{\mathrm{p}}(X) \leq \dim_{\mathrm{p}}(A^{\mathrm{i.o.}}) \leq t$. □

The following corollary is enough to show that certain classes have dimension 0.

Corollary 7.3. *For any* p_i*-printable language* A*,* $\dim_{\mathrm{p}_i}(A^{\mathrm{i.o.}}) = 0$.

Proof. Since every p_i-printable language A is also p_i-rankable and has $H_A = 0$, the corollary follows from Theorem 7.2. □

We now use the p_i-printable corollary to show that appropriately bounded nonuniform complexity classes have dimension 0.

Theorem 7.4. *For all* $c \in \mathbb{N}$*,*

$$\begin{aligned}
&\dim_{\mathrm{p}}(\mathrm{DTIME}(2^{cn})/cn) \\
&= \dim_{\mathrm{p}_2}(\mathrm{DTIME}(2^{n^c})/n^c) \\
&= \dim_{\mathrm{p}_3}(\mathrm{DTIME}(2^{2^{(\log n)^c}})/2^{(\log n)^c}) \\
&= 0.
\end{aligned}$$

Proof. Let $U \in \mathrm{DTIME}(2^{(c+1)n})$ be universal for $\mathrm{DTIME}(2^{cn})$ in the sense that $\mathrm{DTIME}(2^{cn}) = \{U_i \mid i \in \mathbb{N}\}$ where $U_i = \{x \mid \langle i, x\rangle \in U\}$. For each $i \in \mathbb{N}$, define

$$A_i = \{B_{\leq n} \mid (\forall m \leq n)(\exists h_m \in \{0,1\}^{cm}) x \in B_{=m} \iff \langle x, h_m\rangle \in U_i\},$$

where $B_{\leq n}$ represents a characteristic string as in the proof of Theorem 5.3. Let

$$A = \{w \mid (\exists i \leq |w|) w \in A_i\}.$$

Then $\mathrm{DTIME}(2^{cn})/cn \subseteq A^{\mathrm{i.o.}}$. Also, A is p-printable by cycling through all possible advice strings. Therefore $\dim_{\mathrm{p}}(\mathrm{DTIME}(2^{cn})/cn) = 0$ follows from Corollary 7.3.

The p_2- and p_3-dimension statements are proved analogously. □

We now improve Theorem 7.1, replacing measure 0 by dimension 0.

Theorem 7.5. *For all* $c \geq 1$*,*

$$\dim(\mathrm{SIZE}(n^c) \mid \mathrm{EXP}) = \dim_{\mathrm{p}_2}(\mathrm{SIZE}(n^c)) = 0$$

and

$$\dim(\mathrm{P/poly} \mid \mathrm{E}_3) = \dim_{\mathrm{p}_3}(\mathrm{P/poly}) = 0.$$

Proof. Since $\mathrm{SIZE}(s(n)) \subseteq \mathrm{P}/O(s(n)\log s(n))$ for any polynomial $s(n)$, this follows immediately from Theorem 7.4. □

8 Infinitely-Often Classes

For a class $\mathcal{C}$, let

$$\text{io-}\mathcal{C} = \{A \subseteq \{0,1\}^* \mid (\exists B \in \mathcal{C})(\exists^\infty n) A_{=n} = B_{=n}\}$$

be the *io-class of* $\mathcal{C}$. Resource-bounded measure 0 results for nonuniform classes typically also hold for the io-class versions [26]. However, Gu showed the following general lower bound for infinitely-often classes.

Theorem 8.1. (Gu [11]) *For every class $\mathcal{C}$ that contains the empty language $\emptyset$,*

$$\dim_{\mathrm{H}}(\text{io-}\mathcal{C}) \geq \frac{1}{2}.$$

Following a preliminary version of this paper, Gu [11] calculated the dimensions of the infinitely-often versions of the classes in Theorem 7.5 to be exactly $\frac{1}{2}$; that is, the lower bound in Theorem 8.1 is tight for these classes. We will give another proof of this using Theorem 7.2. We first present an infinitely-often version of Theorem 7.4.

Theorem 8.2. *For all $c \in \mathbb{N}$,*

$$\begin{aligned}&\dim_{\mathrm{p}}(\text{io-}[\mathrm{DTIME}(2^{cn})/cn])\\&= \dim_{\mathrm{p}_2}(\text{io-}[\mathrm{DTIME}(2^{n^c})/n^c])\\&= \dim_{\mathrm{p}_3}(\text{io-}[\mathrm{DTIME}(2^{2^{(\log n)^c}})/2^{(\log n)^c}])\\&= \tfrac{1}{2}.\end{aligned}$$

Proof. This is similar to the proof of Theorem 7.4, except here we need to use Theorem 7.2 about rankable entropy rates rather than Corollary 7.3. We focus on the p-dimension case.

Let $U \in \mathrm{DTIME}(2^{(c+1)n})$ be universal for $\mathrm{DTIME}(2^{cn})$ as in the proof of Theorem 7.4. For each $i \in \mathbb{N}$, define

$$A_i = \{B_{\leq n} \mid (\exists h_n \in \{0,1\}^{cn}) x \in B_{=n} \iff \langle x, h_n \rangle \in U_i\}.$$

Let

$$A = \{w \mid (\exists i \leq |w|) w \in A_i\}.$$

Then $\text{io-}[\mathrm{DTIME}(2^{cn})/cn] \subseteq A^{\text{i.o.}}$. Also,

$$\begin{aligned}\log |A_{=2^{n+1}-1}| &\leq \log \left[\left|\{0,1\}^{2^n-1}\right| \cdot (2^{n+1}-1)2^{cn} \right]\\&\leq \log 2^{2^n+(c+1)n+1}\\&= 2^n + (c+1)n + 1,\end{aligned}$$

so $H_A = \frac{1}{2}$ because

$$\limsup_{n\to\infty} \frac{2^n + (c+1)n + 1}{2^{n+1}-1} = \frac{1}{2}.$$

Finally, we claim that A is p-rankable. We need to be able to compute the rank in $A_{=2^{n+1}-1}$ of a given characteristic string $B_{\leq n}$. As in Corollary 7.3, the set

$$C_n = \{w \in \{0,1\}^{2^n} \mid 0^{2^n-1} w \in A\}$$

of suffixes of strings in $A_{=2^{n+1}-1}$ is polynomial-time printable by cycling through all possible advice strings. This makes computing the rank of $B_{\leq n}$ easy: compute the rank of $B_{=n}$ in C_n and add it to $|C_n|$ times the number of lexicographic predecessors of $B_{<n}$. □

We now have another proof of the dimension upper bounds in Gu's aforementioned theorem. (Gu's proof used relationships between Kolmogorov complexity and circuit-size complexity [2, 3].)

Theorem 8.3. (Gu [11]) *For all $c \geq 1$,*

$$\dim(\text{io-SIZE}(n^c) \mid \mathrm{EXP}) = \dim_{\mathrm{p}_2}(\text{io-SIZE}(n^c)) = \frac{1}{2}$$

and

$$\dim(\text{io-}[\mathrm{P/poly}] \mid \mathrm{E}_3) = \dim_{\mathrm{p}_3}(\text{io-}[\mathrm{P/poly}]) = \frac{1}{2}.$$

Proof. Since io-SIZE$(s(n)) \subseteq$ io-$[\mathrm{P}/O(s(n)\log s(n))]$ for any polynomial $s(n)$, the dimension upper bounds follows immediately from Theorem 8.2. The lower bounds follow from Theorem 8.1. $\square$

Next we turn our attention to the infinitely-often versions of the exponential-size circuit-complexity classes we studied earlier. We have the following in comparison to Theorem 5.6.

Theorem 8.4. *For every $\alpha \in [0,1]$,*

$$\dim_{\Delta_3^{\mathrm{P}}}\left(\text{io-SIZE}\left(\alpha\frac{2^n}{n}\right)\right) = \frac{1+\alpha}{2}.$$

This theorem is immediate from Lemmas 8.5 and 8.6 below, using Theorem 5.5 to establish the upper bound.

Lemma 8.5. *For every $\alpha \in [0,1]$,*

$$\mathcal{H}_{\mathrm{NP}}\left(\text{io-SIZE}\left(\alpha\frac{2^n}{n}\right)\right) \leq \frac{1+\alpha}{2}.$$

Proof. Define

$$A = \left\{B_{\leq n} \,\middle|\, B_{=n} \text{ has a circuit of size} \leq \alpha\tfrac{2^n}{n}\ \right\}.$$

Then $A \in \mathrm{NP}$ and io-SIZE$(\alpha\frac{2^n}{n}) \subseteq A^{\text{i.o.}}$. A calculation similar to the one in Theorem 5.3 shows that $H_A = \frac{1+\alpha}{2}$. $\square$

Lemma 8.6. *For every $\alpha \in [0,1]$,*

$$\dim_{\mathrm{H}}\left(\text{io-SIZE}\left(\alpha\frac{2^n}{n}\right)\right) \geq \frac{1+\alpha}{2}.$$

Proof. This proof is similar to the proof of Theorem 5.8 and is also inspired by the same technique of Gu [12].

Let $s(n) = \alpha\frac{2^n}{n}$ and let $s < t < \alpha$. Let $r = \frac{1+s}{2}$ and let d be an arbitrary r-gale. It suffices to show that io-SIZE$(s(n)) \not\subseteq S^\infty[d]$.

We define a language A inductively. Let $A_{\leq 1} = \emptyset$. Assume that $A_{\leq n}$ has been defined. We will extend this to define $A_{\leq 2n}$. Let w be the characteristic string of $A_{\leq n}$. As in the proof of Theorem 5.8, define u of length $2^{2n} - 2^{n+1} + \lceil t2^{2n}\rceil$ so that $d(wu') \leq 2^{(r-1)|u'|}d(w)$ for all $u' \sqsubseteq u$. Let $v = 0^{2^{2n} - \lceil t2^{2n}\rceil}$ and let $A_{\leq 2n}$ have characteristic string wuv. For all $v' \sqsubseteq v$,

$$\begin{aligned} d(wuv') \leq 2^{s|uv'|-|u|}d(w) &\leq 2^{r(2^{2n+1}-2^{n+1})-|u|}d(w) \\ &= 2^{(1+s)2^{2n}-r2^{n+1}-2^{2n}+2^{n+1}-\lceil t2^{2n}\rceil}d(w) \\ &\leq 2^{s2^{2n}+2^{n+1}-\lceil t2^{2n}\rceil}d(w). \end{aligned}$$

When n is sufficiently large, this last multiplier is less than 1. It follows that d is bounded on A, so $A \notin S^\infty[d]$. Also, arguing as in the proof Theorem 5.8, $A_{=n}$ has a circuit of size at most $s(n)$ whenever n is a sufficiently large power of 2, so $A \in$ io-SIZE$(s(n))$. $\square$

9 Superranking and Kolmogorov Complexity

For many sets for which the p-dimension has been calculated it can be shown that an equality actually holds in Theorem 7.2. In this section we show that we always get an equality when a generalization of ranking is used.

9.1 Superranking

Definition. Let $A \subseteq \{0,1\}^*$.

1. A *superranking function* for A is a function $f : \{0,1\}^* \to \mathbb{N}$ that is nondecreasing (i.e., $f(x) \le f(x+1)$ for all x) and satisfies $f(x) > f(x-1)$ for all $x \in A$.

2. The *rate* of a superranking function f is

$$H_f = \limsup_{n\to\infty} \frac{\log[f(1^n) - f(1^{n-1})]}{n}.$$

3. The *polynomial-time superranking rate of* A is

$$H_A^* = \inf\{H_f \mid f \in \text{p is a superranking function for } A\ \}.$$

Intuitively, a superranking function f for A is an overestimate of the ranking function of A. It always increases when rank_A increases, but may increase by an amount larger than 1 and may increase on strings that are not in A.

The quantity $f(1^n) - f(1^{n-1})$ is an upper bound on $|A_{=n}|$. For this reason, we have $H_A \le H_A^* \le 1$ for any language A. If A is p-rankable, then $H_A = H_A^*$ because rank_A is a polynomial-time superranking function for A and $H_{\text{rank}_A} = H_A$.

We now use superranking rates to define a variation of the P-entropy rate.

Definition. For any $X \subseteq \mathbf{C}$, define

$$\mathcal{H}_{\text{P}}^*(X) = \inf\{H_A^* \mid A \in \text{P and } X \subseteq A^{\text{i.o.}}\}.$$

From our observations above, it is clear that

$$\mathcal{H}_{\text{P}}(X) \le \mathcal{H}_{\text{P}}^*(X) \le \mathcal{H}_{\text{p-rankable}}(X)$$

for all $X \subseteq \mathbf{C}$. We now show that $\mathcal{H}_{\text{P}}^*$ is exactly the same as $\dim_{\text{p}}$. Note that this improves Theorem 7.2.

Theorem 9.1. *For any* $X \subseteq \mathbf{C}$*,*

$$\dim_{\text{p}}(X) = \mathcal{H}_{\text{P}}^*(X).$$

Proof. The proof that $\dim_{\text{p}}(X) \le \mathcal{H}_{\text{P}}^*(X)$ is a modification of the proof of Theorem 7.2. Let $t > s > \mathcal{H}_{\text{P}}^*(X)$ with $2^s \in \mathbb{Q}$ and take an $A \in \text{P}$ such that $X \subseteq A^{\text{i.o.}}$ and $H_A^* < s$. Then let f be a superranking function for A that satisfies $H_f < s$. Now for any w and n, we can upper bound $\text{ext}_A(w,n)$ by $f(w1^{n-|w|}) - f(w0^{n-|w|} - 1)$. Define the measure $\rho_n(w)$ using this upper bound instead of $\text{ext}_A(w,n)$. Then for any $w \in A$ we have $\rho_{|w|}(w) = [f(w) - f(w-1)]2^{-s|w|} \ge 2^{-s|w|}$. The rest of the proof goes through to show that $\dim_{\text{p}}(X) \le t$.

For the other inequality, let $s > \dim_{\text{p}}(X)$ such that 2^s is rational. It suffices to show that $\mathcal{H}_{\text{P}}^*(X) \le s$. Let μ be an exactly polynomial-time computable measure such that for all $S \in X$,

$$\liminf_{n\to\infty} \frac{-\log \mu(S\upharpoonright n)}{n} < s.$$

We can assume without loss of generality that $\mu(\lambda) = 1$. Letting

$$A = \{w \mid \mu(w) \ge 2^{-sn}\},$$

we have $X \subseteq A^{\text{i.o.}}$. Define $f : \{0,1\}^* \to \mathbb{N}$ by

$$f(w) = \left\lceil 2^{s|w|} \sum_{\substack{|x|=|w| \\ x \leq w}} \mu(x) \right\rceil + f(1^{|w|-1}).$$

Then f is a superranking function for A. For all n, $f(1^n) - f(1^{n-1}) = \lceil 2^{sn} \rceil$, so $H_f \leq s$. Now we will show that f is polynomial-time computable. Let $I_w = \{(w \upharpoonright i)0 \mid w[i] = 1\}$. Then $x < w$ if and only if x has a prefix in I_w. Using the additivity property of μ, we have

$$\sum_{\substack{|x|=|w| \\ x<w}} \mu(x) = \sum_{y \in I_w} \sum_{\substack{|x|=|w| \\ y \sqsubseteq x}} \mu(x) = \sum_{y \in I_w} \mu(y).$$

Given $f(1^{|w|-1})$, we can therefore compute $f(w)$ using at most $|w|+1$ evaluations of μ on strings no longer than w. This shows that f is polynomial-time computable. Therefore $\mathcal{H}^*_{\mathrm{P}}(X) \leq H^*_A \leq H_f \leq s$. □

We can now give a hypothesis that implies $\mathcal{H}_{\mathrm{P}}$ is equal to $\dim_{\mathrm{p}}$. The plausibility of the hypothesis is not clear.

Corollary 9.2. *If $H_A = H^*_A$ for every $A \in \mathrm{P}$, then $\dim_{\mathrm{p}}(X) = \mathcal{H}_{\mathrm{P}}(X)$ for all $X \subseteq \mathbf{C}$.*

9.2 Kolmogorov Complexity

For a function $r : \mathbb{N} \to \mathbb{N}$ and a string x, let $K(x)$ be the Kolmogorov complexity of x, let $K^r(x)$ be the r-time-bounded Kolmogorov complexity of x, and let $KS^r(x)$ be the r-space-bounded Kolmogorov complexity of x. (Here $K^r(x)$ is the minimum length of a program that causes a universal Turing machine to output x in at most $r(|x|)$ time, and $KS^r(x)$ is defined analogously. Because we will be dividing by $|x|$ in what follows, it is makes no difference if we use plain complexity or prefix-free complexity.) For a sequence $S \in \mathbf{C}$, define

$$\mathcal{K}(S) = \liminf_{n\to\infty} \frac{K(S\upharpoonright n)}{n}, \quad \mathcal{KS}^r(S) = \liminf_{n\to\infty} \frac{KS^r(S\upharpoonright n)}{n}, \text{ and } \mathcal{K}^r(S) = \liminf_{n\to\infty} \frac{K^r(S\upharpoonright n)}{n}.$$

For any $X \subseteq \mathbf{C}$, define

$$\mathcal{K}(X) = \sup_{S\in X} \mathcal{K}(S), \quad \mathcal{KS}^r(X) = \sup_{S\in X} \mathcal{KS}^r(S), \text{ and } \mathcal{K}^r(X) = \sup_{S\in X} \mathcal{K}^r(S).$$

Let poly and comp be the classes of all functions mapping $\mathbb{N}$ to $\mathbb{N}$ that are polynomially-bounded and computable, respectively. For any $X \subseteq \mathbf{C}$, define

$$\mathcal{K}^{\text{poly}}(X) = \inf_{p\in\text{poly}} \mathcal{K}^p(X), \quad \mathcal{KS}^{\text{poly}}(X) = \inf_{p\in\text{poly}} \mathcal{KS}^p(X), \text{ and } \mathcal{KS}^{\text{comp}}(X) = \inf_{r\in\text{comp}} \mathcal{KS}^r(X).$$

Mayordomo [33], building on [30], showed that constructive dimension can be equivalently defined using Kolmogorov complexity.

Theorem 9.3. (Mayordomo [33]) *For any $X \subseteq \mathbf{C}$, $\text{cdim}(X) = \mathcal{K}(X)$.*

This can be extended to the computable and polynomial-space dimensions by imposing computable and polynomial-space constraints on the Kolmogorov complexity.

Theorem 9.4. (Hitchcock [14]) *For any* $X \subseteq \mathbf{C}$*,* $\dim_{\mathrm{comp}}(X) = \mathcal{KS}^{\mathrm{comp}}(X)$ *and* $\dim_{\mathrm{pspace}}(X) = \mathcal{KS}^{\mathrm{poly}}(X)$*.*

It is unknown if $\dim_{\mathrm{p}}(X) = \mathcal{K}^{\mathrm{poly}}(X)$ holds for all X. We can use our superranking characterization of p-dimension to show that one inequality always holds. The following proposition shows that strings in a language A have polynomial-time Kolmogorov complexity that is not much more than the polynomial-time superranking rate of A.

Proposition 9.5. *Let* $A \subseteq \{0,1\}^*$ *and let* $s > H_A^*$*. Then there is a polynomial* p *such that for all but finitely many* $x \in A$*,* $K^p(x) \leq s|x|$*.*

Proof. Let $s > r > H_A^*$ and let f be a polynomial-time computable superranking function for A that satisfies $f(1^n) \leq 2^{rn}$ for all sufficiently large n. Then for any $x \in A$ with $|x|$ large enough, $f(x)$ can be represented as a binary string of length at most $r|x|$. Given $f(x)$, we can use binary search to find x. Therefore $K^p(x) \leq r|x| + c \leq s|x|$ holds for all but finitely many $x \in A$, where p is some polynomial and c is some constant. □

We can now sandwich $\mathcal{K}^{\mathrm{poly}}$ between the NP-entropy rate and p-dimension.

Theorem 9.6. *For any* $X \subseteq \mathbf{C}$*,*

$$\mathcal{H}_{\mathrm{NP}}(X) \leq \mathcal{K}^{\mathrm{poly}}(X) \leq \dim_{\mathrm{p}}(X).$$

Proof. Let $s > \dim_{\mathrm{p}}(X)$. By Theorem 9.1, let $A \in \mathrm{P}$ such that $X \subseteq A^{\mathrm{i.o.}}$ and $H_A^* < s$. It follows from Proposition 9.5 that $\mathcal{K}^{\mathrm{poly}}(X) \leq \mathcal{K}^{\mathrm{poly}}(A^{\mathrm{i.o.}}) \leq s$. Therefore $\mathcal{K}^{\mathrm{poly}}(X) \leq \dim_{\mathrm{p}}(X)$.

Now let $s > \mathcal{K}^{\mathrm{poly}}(X)$ be rational and let p be a polynomial such that $\mathcal{K}^p(S) < s$ for all $S \in X$. Then the language

$$A = \{x \mid K^p(x) \leq s|x|\}$$

is in NP and satisfies $X \subseteq A^{\mathrm{i.o.}}$. Since $|A_{=n}| \leq 2^{sn+1}$ for all n, we have $H_A \leq s$, so $\mathcal{H}_{\mathrm{NP}}(X) \leq s$. Therefore $\mathcal{H}_{\mathrm{NP}}(X) \leq \mathcal{K}^{\mathrm{poly}}(X)$. □

10 Conclusion

We have given several new relationships between resource-bounded dimension, entropy rates, and compression. Now we know that for any $X \subseteq \mathbf{C}$,

$$\left\{ \begin{array}{c} \dim_{\mathrm{pspace}}(X) \\ = \\ \mathcal{H}_{\mathrm{PSPACE}}(X) \\ = \\ \mathcal{KS}^{\mathrm{poly}}(X) \end{array} \right\} \leq \dim_{\Delta_3^{\mathrm{p}}}(X) \leq \mathcal{H}_{\mathrm{NP}}(X) \leq \left\{ \begin{array}{c} \mathcal{H}_{\mathrm{P}}(X), \\ \mathcal{K}^{\mathrm{poly}}(X) \end{array} \right\} \leq \left\{ \begin{array}{c} \dim_{\mathrm{p}}(X) \\ = \\ \mathcal{H}_{\mathrm{P}}^*(X) \end{array} \right\} \leq \mathcal{H}_{\text{p-rankable}}(X).$$

(We do not know of any relationship between $\mathcal{H}_{\mathrm{P}}$ and $\mathcal{K}^{\mathrm{poly}}$.) These results were useful for improving previous results about the resource-bounded measure and dimension of circuit-size complexity classes, and we anticipate that these general tools we have developed will be useful in future work.

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