Gales Suffice for Constructive Dimension *

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Abstract

Supergales, generalizations of supermartingales, have been used by Lutz (2002) to define the constructive dimensions of individual binary sequences. Here it is shown that gales, the corresponding generalizations of martingales, can be equivalently used to define constructive dimension.

1 Introduction

Effective martingales have been very useful objects in theoretical computer science. Schnorr [7, 8] used constructive martingales to give an equivalent definition of Martin-Löf randomness [6]. Martingales computable within resource bounds have been used by Lutz [3] to define various resource-bounded measures that have been successful in complexity theory. In all these cases, it is known that replacing the constructive or resource-bounded martingales with constructive or resource-bounded supermartingales results in an equivalent definition.

Lutz [4] recently introduced supergales and gales as natural generalizations of supermartingales and martingales, respectively. He showed that gales can be used to characterize classical Hausdorff dimension. With this as a motivation, Lutz used gales computable within resource bounds to define resource-bounded dimensions that work inside of complexity classes. He also showed that supergales may be used in place of gales to give equivalent definitions of these dimensions.

Constructive dimension [2] refines the theory of Martin-Löf randomness by assigning each individual binary sequence a dimension. Lutz used constructive supergales to define constructive dimension. Supergales were used rather than gales because he was able to show that optimal constructive supergales exist. The questions of whether optimal constructive gales exist and whether gales can be used to equivalently define constructive dimension were left open.

Therefore, martingales and supermartingales are known to give equivalent definitions for all the applications mentioned above, and gales and supergales are known to give equivalent definitions for all the applications mentioned above *except* constructive dimension. Here it is shown that constructive gales give an equivalent definition of constructive dimension. The proof is a simple and direct construction that uses some ideas from an earlier paper by the author [1]. As a corollary we obtain a form of optimal constructive gales.

2 Preliminaries

The set of natural numbers is $\mathbb{N} = \{0, 1, 2, \ldots\}$. The set of binary strings of length $n \in \mathbb{N}$ is $\{0,1\}^n$. The set of all finite binary strings is $\{0,1\}^*$. The empty string is λ. For a language $A \subseteq \{0,1\}^*$, we write $A_{=n}$ for

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the set of strings in A of length n. For strings $w, v \in \{0,1\}^*$, we write $w \sqsubseteq v$ if w is a prefix of v. $\mathbf{C}$ is the Cantor space of all infinite binary sequences. For a sequence $S \in \mathbf{C}$, $S[0..n-1]$ is the prefix of S of length n.

A real number r is computable if there is a computable function $f : \mathbb{N} \to \mathbb{Q}$ such that $|f(n) - r| \leq 2^{-n}$ for all $n \in \mathbb{N}$. A function $g : \{0,1\}^* \to [0, \infty)$ is constructive if there is a computable function $h : \{0,1\}^* \times \mathbb{N} \to \mathbb{Q}$ such that for all $w \in \{0,1\}^*$, $h(w,n) \leq h(w, n+1) < g(w)$ for all $n \in \mathbb{N}$ and $g(w) = \sup_{n \in \mathbb{N}} h(w,n)$.

3 Constructive Dimension

Constructive dimension was introduced by Lutz [2]. Here we review the basic concepts. We begin by defining supergales and gales.

Definition. Let $s \in [0, \infty)$. A function $d : \{0,1\}^* \to [0, \infty)$ is an *s-supergale* if

$$d(w) \geq \frac{d(w0) + d(w1)}{2^s} \tag{3.1}$$

for all $w \in \{0,1\}^*$. If equality holds in (3.1) for all strings w, then d is an *s-gale*.

Note that 1-gales are martingales and 1-supergales are supermartingales. We are particularly interested in the success sets of supergales and gales.

Definition. The *success set* of a supergale $d : \{0,1\}^* \to [0, \infty)$ is

$$S^\infty[d] = \left\{ S \in \mathbf{C} \,\middle|\, \limsup_{n \to \infty} d(S[0..n-1]) = \infty \right\}.$$

Notation. For any $X \subseteq \mathbf{C}$, we define the sets of nonnegative real numbers

$$\mathcal{G}_{\text{constr}}(X) = \left\{ s \,\middle|\, \begin{array}{l} \text{there exists a constructive} \\ s\text{-gale } d \text{ for which } X \subseteq S^\infty[d] \end{array} \right\}$$

and

$$\widehat{\mathcal{G}}_{\text{constr}}(X) = \left\{ s \,\middle|\, \begin{array}{l} \text{there exists a constructive} \\ s\text{-supergale } d \text{ for which } X \subseteq S^\infty[d] \end{array} \right\}.$$

Constructive dimension is defined in terms of succeeding constructive supergales.

Definition. For a set $X \subseteq \mathbf{C}$, the *constructive dimension* of X is

$$\text{cdim}(X) = \inf \widehat{\mathcal{G}}_{\text{constr}}(X).$$

For a sequence $S \in \mathbf{C}$, the *constructive dimension* of S is

$$\text{cdim}(S) = \text{cdim}(\{S\}).$$

We now define two notions of optimality for a class of supergales.

Definition. Let d^* be a supergale and let $\mathcal{D}$ be a class of supergales.

1. We say that d^* is *multiplicatively optimal* for $\mathcal{D}$ if for each $d \in \mathcal{D}$ there is an $\alpha > 0$ such that $d^*(w) \geq \alpha d(w)$ for all $w \in \{0,1\}^*$.

2. We say that d^* is *successively optimal* for $\mathcal{D}$ if for every $d \in \mathcal{D}$, $S^\infty[d] \subseteq S^\infty[d^*]$.

Lutz used Levin's universal constructive semimeasure [9] to show that there exist multiplicatively optimal supergales.

Theorem 3.1. (Lutz [2])*For any computable $s \in [0,\infty)$ there is a constructive s-supergale $\mathbf{d}^{(s)}$ that is multiplicatively optimal for the class of constructive s-supergales.*

Theorem 3.1 was used to prove the following cornerstone of constructive dimension theory.

Theorem 3.2. (Lutz [2])*For any $X \subseteq \mathbf{C}$,*

$$\mathrm{cdim}(X) = \sup_{S\in X} \mathrm{cdim}(S).$$

Remark. In [5], a conference paper preceding [2], Lutz defined constructive dimension using constructive gales. There Lutz used a incorrect assertion about martingales to argue that for each computable s there exists a constructive s-gale that is multiplicatively optimal for the class of constructive s-gales. These "optimal gales" were then used to prove Theorem 3.2. These flawed arguments were subsequently noticed and corrected in [2] by reformulating constructive dimension in terms of constructive supergales. The multiplicatively optimal supergales of Theorem 3.1 exist and Theorem 3.2 is true in the reformulation. However, Lutz left open the questions of whether there exist optimal constructive gales and whether constructive dimension can be equivalently defined using constructive gales. This paper addresses these questions.

4 The Strength of Gales

Theorem 4.1. *Let $0 \le r < t$ be computable real numbers. Then for any constructive r-supergale d, there exists a constructive t-gale d' such that $S^\infty[d] \subseteq S^\infty[d']$.*

Proof. Let d be a constructive r-supergale and assume without loss of generality that $d(\lambda) < 1$. Define the language $A = \{w \in \{0,1\}^* | d(w) > 1\}$. Observe that A is computably enumerable. For all $n \in \mathbb{N}$, $\sum_{w\in\{0,1\}^n} d(w) \le 2^{rn}$, so $|A_{=n}| \le 2^{rn}$.

For each $n \in \mathbb{N}$, define a function $d'_n : \{0,1\}^* \to [0,\infty)$ by

$$d'_n(w) = \begin{cases} 2^{-t(n-|w|)} \cdot \left|\{v \in A_{=n} | w \sqsubseteq v\}\right| & \text{if } |w| \le n \\ 2^{(t-1)(|w|-n)} d'_n(w[0..n-1]) & \text{if } |w| > n. \end{cases}$$

Then for all n, d'_n is a t-gale and $d'_n(w) = 1$ for all $w \in A_{=n}$.

Let $s \in (r,t)$ be computable and define a function d' on $\{0,1\}^*$ by $d' = \sum_{n=0}^\infty 2^{(s-r)n} d'_n$. Then

$$d'(\lambda) = \sum_{n=0}^\infty 2^{(s-r)n} 2^{-tn} |A_{=n}| \le \sum_{n=0}^\infty 2^{(s-t)n} < \infty,$$

and it follows by induction that $d'(w) < \infty$ for all strings w. Therefore, by linearity, d' is a t-gale. Also, because the language A is computably enumerable, d' is constructive.

Let $S \in S^\infty[d]$. Then for infinitely many $n \in \mathbb{N}$, $S[0..n-1] \in A$. For each of these n,

$$d'(S[0..n-1]) \ge 2^{(s-r)n} d'_n(S[0..n-1]) = 2^{(s-r)n},$$

so $S \in S^\infty[d']$. Therefore $S^\infty[d] \subseteq S^\infty[d']$. □

Constructive dimension may now be equivalently defined using gales instead of supergales.

Theorem 4.2. *For all $X \subseteq \mathbf{C}$, $\mathrm{cdim}(X) = \inf \mathcal{G}_{\mathrm{constr}}(X)$.*

Proof. Because any gale is also a supergale, $\mathcal{G}_{\mathrm{constr}}(X) \subseteq \widehat{\mathcal{G}}_{\mathrm{constr}}(X)$, so $\mathrm{cdim}(X) = \inf \widehat{\mathcal{G}}_{\mathrm{constr}}(X) \le \inf \mathcal{G}_{\mathrm{constr}}(X)$ is immediate.

Let $t > r > \mathrm{cdim}(X)$ be computable real numbers and let d be a constructive r-supergale such that $X \subseteq S^\infty[d]$. By Theorem 4.1, there is a constructive t-gale d' such that $X \subseteq S^\infty[d] \subseteq S^\infty[d']$, so $t \in \mathcal{G}_{\mathrm{constr}}(X)$. As this holds for any computable $t > \mathrm{cdim}(X)$, we have $\inf \mathcal{G}_{\mathrm{constr}}(X) \le \mathrm{cdim}(X)$. □

We can also state the existence of a form of optimal constructive gales.

Corollary 4.3. *For all computable real numbers $t > r \geq 0$ there exists a constructive t-gale that is successively optimal for the class of constructive r-supergales.*

Proof. Let $\mathbf{d}^{(r)}$ be the constructive r-supergale from Theorem 3.1 that is multiplicatively optimal for the constructive r-supergales. Theorem 4.1 provides a constructive t-gale d' that succeeds everywhere that $\mathbf{d}^{(r)}$ does. Therefore $S^\infty[d] \subseteq S^\infty[\mathbf{d}^{(r)}] \subseteq S^\infty[d']$ for any constructive r-supergale d, so the corollary is proved. □

The optimal gales provided by Corollary 4.3 may not be technically as strong as possible, in two respects.

1. Lutz's optimal constructive r-supergale is multiplicatively optimal, whereas our optimal constructive t-gale is only successively optimal. Does there exist a constructive t-gale that is multiplicatively optimal for the class of constructive r-supergales?

2. Our proof seems to require the hypothesis $t > r$. Does there exist a constructive r-gale that is successively optimal for the class of constructive r-supergales?

However, the optimality in Corollary 4.3 remains strong enough to prove Theorem 3.2.

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