MAX3SAT Is Exponentially Hard to Approximate If NP Has Positive Dimension*

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Abstract

Under the hypothesis that NP has positive p-dimension, we prove that any approximation algorithm $\mathcal{A}$ for MAX3SAT must satisfy at least one of the following:

1. For some $\delta > 0$, $\mathcal{A}$ uses at least 2^{n^δ} time.
2. For all $\epsilon > 0$, $\mathcal{A}$ has performance ratio less than $\frac{7}{8} + \epsilon$ on an exponentially dense set of satisfiable instances.

As a corollary, this solves one of Lutz and Mayordomo's "Twelve Problems on Resource-Bounded Measure" (1999).

1 Introduction

MAX3SAT is a well-studied optimization problem. Tight bounds on its polynomial-time approximability are known:

1. There exists a polynomial-time $\frac{7}{8}$-approximation algorithm (Karloff and Zwick [5, 3]). [1]
2. If P $\neq$ NP, then for all $\epsilon > 0$, there does not exist a polynomial-time $(\frac{7}{8} + \epsilon)$-approximation algorithm (Håstad [4]).

Recently there has been some investigation of approximating MAX3SAT in exponential time. For example, for any $\epsilon \in (0, \frac{1}{8}]$, Dantsin, Gavrilovich, Hirsch, and Konev [2] give a $(\frac{7}{8} + \epsilon)$-approximation algorithm for MAX3SAT running in time $2^{8\epsilon k}$ where k is the number of clauses in a formula.

Given these results, it is natural to ask for stronger lower bounds on computation time for MAX3SAT approximation algorithms that have performance ratio greater than $\frac{7}{8}$. Such lower bounds are not known to follow from the hypothesis P $\neq$ NP. In this note we address this question using a stronger hypothesis involving resource-bounded dimension.

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[1] An algorithm with conjectured performance ratio $\frac{7}{8}$ was given in [5], and this conjecture has since been proved according to [3].

About a decade ago, Lutz [6] presented resource-bounded measure as an analogue for classical Lebesgue measure in complexity theory. Resource-bounded measure provides strong, reasonable hypotheses which seem to have more explanatory power than weaker, traditional complexity-theoretic hypotheses. The hypothesis that NP does not have p-measure 0, $\mu_{\rm p}(\rm NP) \neq 0$, implies $\rm P \neq NP$ and is known to have many plausible consequences that are not known to follow from $\rm P \neq NP$.

Resource-bounded dimension was recently introduced by Lutz [7] as an analogue of classical Hausdorff dimension for complexity theory. Resource-bounded dimension refines resource-bounded measure by providing a spectrum of weaker, but still strong, hypotheses. We will use the hypothesis that NP has positive p-dimension, $\dim_{\rm p}(\rm NP) > 0$. This hypothesis is implied by $\mu_{\rm p}(\rm NP) \neq 0$ and implies $\rm P \neq NP$.

Under the hypothesis $\dim_{\rm p}(\rm NP) > 0$, we give an exponential-time lower bound for approximating MAX3SAT beyond the known polynomial-time achievable ratio of $\frac{7}{8}$ on all but a subexponentially-dense set of satisfiable instances. Put another way, we prove:

> If $\dim_{\rm p}(\rm NP) > 0$, then any approximation algorithm $\mathcal{A}$ for MAX3SAT must satisfy at least one of the following:
>
> 1. For some $\delta > 0$, $\mathcal{A}$ uses at least 2^{n^δ} time.
> 2. For all $\epsilon > 0$, $\mathcal{A}$ has performance ratio less than $\frac{7}{8} + \epsilon$ on an exponentially dense set of satisfiable instances.

Lutz and Mayordomo asked whether the hypothesis $\mu_{\rm p}(\rm NP) \neq 0$ implies an exponential-time lower bound on approximation schemes for MAXSAT [8]. Our result gives a strong affirmative answer to this question: we obtain a stronger conclusion from the weaker $\dim_{\rm p}(\rm NP) > 0$ hypothesis. In fact, after we present the dimension result, we give an easy proposition that achieves an exponential-time lower bound from a hypothesis even weaker than $\dim_{\rm p}(\rm NP) > 0$.

In section 2 we give our notation and basic definitions. Resource-bounded measure and dimension are briefly reviewed in section 3. Section 4 contains a dimension result used in proving our main theorem. The main theorem is proved in section 5. Section 6 concludes by summarizing the inapproximability results for MAX3SAT under strong hypotheses.

2 Preliminaries

The set of all finite binary strings is $\{0,1\}^*$. We use the standard enumeration of binary strings $s_0 = \lambda, s_1 = 0, s_2 = 1, s_3 = 00, \ldots$. The length of a string $x \in \{0,1\}^*$ is denoted by $|x|$.

All *languages* (decision problems) in this paper are encoded as subsets of $\{0,1\}^*$. For a language $A \subseteq \{0,1\}^*$, we define $A_{\leq n} = \{x \in A \big| |x| \leq n\}$. We write $A[0..n-1]$ for the n-bit prefix of the characteristic sequence of A according to the standard enumeration of strings.

We say that a language A is *(exponentially) dense* if there is an $\alpha > 0$ such that $|A_{\leq n}| > 2^{n^\alpha}$ holds for all but finitely many n. We write DENSE for the class of all dense languages.

For any classes $\mathcal{C}$ and $\mathcal{D}$ of languages we define the classes

$$\mathcal{C} \uplus \mathcal{D} = \{A \cup B \,|\, A \in \mathcal{C}, B \in \mathcal{D}\}$$

and

$$\mathrm{P_m}(\mathcal{C}) = \left\{A \subseteq \{0,1\}^* \,\middle|\, (\exists B \in \mathcal{C}) A \leq_{\rm m}^{\rm P} B\right\}.$$

A real-valued function $f : \{0,1\}^* \to [0,\infty)$ is polynomial-time computable if there exists a polynomial-time computable function $g : \mathbb{N} \times \{0,1\}^* \to [0,\infty) \cap \mathbb{Q}$ such that

$$|f(x) - g(n,x)| \leq 2^{-n}$$

for all $x \in \{0,1\}^*$ and $n \in \mathbb{N}$ where n is represented in unary.

For an instance x of 3SAT we write MAX3SAT(x) for the maximum fraction of clauses of x that can be satisfied by a single assignment.

An *approximation algorithm* $\mathcal{A}$ for MAX3SAT outputs an assignment of the variables for each instance of 3SAT. For each instance x we write $\mathcal{A}(x)$ for the fraction of clauses satisfied by the assignment produced by $\mathcal{A}$ for x.

An approximation algorithm $\mathcal{A}$ has *performance ratio* α on x if $\mathcal{A}(x) \geq \alpha \cdot \mathrm{MAX3SAT}(x)$. If $\mathcal{A}$ has performance ratio α on all instances, then $\mathcal{A}$ is an *α-approximation algorithm.*

Håstad proved the following in order to show that satisfiable instances of 3SAT cannot be distinguished from instances x with $\mathrm{MAX3SAT}(x) < \frac{7}{8} + \epsilon$ in polynomial-time unless P=NP.

Theorem 2.1 (Håstad [4]) *For each $\epsilon > 0$, there exists a polynomial-time computable function f_ϵ such that for all $x \in \{0,1\}^*$,*

$$\begin{aligned} x \in \mathrm{SAT} &\Rightarrow \mathrm{MAX3SAT}(f_\epsilon(x)) = 1 \\ x \notin \mathrm{SAT} &\Rightarrow \mathrm{MAX3SAT}(f_\epsilon(x)) < \tfrac{7}{8} + \epsilon. \end{aligned}$$

We will use the functions f_ϵ from Theorem 2.1 later in the paper.

3 Resource-Bounded Measure and Dimension

In this section we review enough resource-bounded measure and dimension to present our result. Full details of these theories are available in Lutz's introductory papers [6, 7].

Definition 3.1 *Let $s \in [0,\infty)$.*

1. *A function $d : \{0,1\}^* \to [0,\infty)$ is an s-*gale *if for all $w \in \{0,1\}^*$,*

$$d(w) = \frac{d(w0) + d(w1)}{2^s}.$$

2. *A* martingale *is a 1-gale.*

Intuitively, a gale is viewed as a function betting on an unknown binary sequence. If w is a prefix of the sequence, then the capital of the gale after placing its first $|w|$ bets is given by $d(w)$. Assuming that w is a prefix of the sequence, the gale places bets on $w0$ and $w1$ also being prefixes. The parameter s determines the fairness of the betting; as s decreases the betting is less fair. The goal of a gale is to bet successfully on languages.

Definition 3.2 *Let $s \in [0,\infty)$ and let d be an s-gale.*

1. *We say d* succeeds on *a language A if*

$$\limsup_{n\to\infty} d(A[0..n-1]) = \infty.$$

2. The success set *of d is*

$$S^\infty[d] = \{A \subseteq \{0,1\}^* | d \text{ succeeds on } A\}.$$

Measure and dimension are defined in terms of succeeding martingales and gales, respectively.

Definition 3.3 *Let $\mathcal{C}$ be a class of languages.*

1. *$\mathcal{C}$ has p-measure 0, written $\mu_{\mathrm{p}}(\mathcal{C}) = 0$, if there exists a polynomial-time martingale d with $\mathcal{C} \subseteq S^\infty[d]$.*
2. *The p-dimension of $\mathcal{C}$ is*

$$\dim_{\mathrm{p}}(\mathcal{C}) = \inf\left\{ s \;\middle|\; \begin{array}{l} \text{there exists a polynomial-time} \\ s\text{-gale } d \text{ for which } \mathcal{C} \subseteq S^\infty[d] \end{array} \right\}.$$

For any class $\mathcal{C}$, $\dim_{\mathrm{p}}(\mathcal{C}) \in [0,1]$. We are interested in hypotheses on the p-dimension and p-measure of NP. The following implications are easy to verify.

$$\begin{aligned} \mu_{\mathrm{p}}(\mathrm{NP}) \neq 0 \;\; &\Rightarrow \;\; \dim_{\mathrm{p}}(\mathrm{NP}) = 1 \\ &\Rightarrow \;\; \dim_{\mathrm{p}}(\mathrm{NP}) > 0 \\ &\Rightarrow \;\; \mathrm{P} \neq \mathrm{NP}. \end{aligned}$$

The following simple lemma will be useful in proving our main result.

Lemma 3.4 *Let $\mathcal{C}$ be a class of languages and $c \in \mathbb{N}$.*

(1) *If $\mu_{\mathrm{p}}(\mathcal{C}) = 0$, then $\mu_{\mathrm{p}}(\mathcal{C} \uplus \mathrm{DTIME}(2^{cn})) = 0$.*

(2) $\dim_{\mathrm{p}}(\mathcal{C} \uplus \mathrm{DTIME}(2^{cn})) = \dim_{\mathrm{p}}(\mathcal{C})$.

Proof: Let $s \in [0,1]$ be such that 2^s is rational and assume that there is a polynomial-time s-gale d succeeding on $\mathcal{C}$. By the Exact Computation Lemma of [7], we may assume that d is exactly computable in polynomial-time. It suffices to give a polynomial-time s-gale succeeding on $\mathcal{C} \uplus \mathrm{DTIME}(2^{cn})$. Let $M_0, M_1, \ldots$ be a standard enumeration of all Turing machines running in time 2^{cn}. Define for each $i \in \mathbb{N}$ and $w \in \{0,1\}^*$,

$$\begin{aligned} d_i(w1) &= \begin{cases} 2^s d_i(w) & \text{if } M_i \text{ accepts } s_{|w|} \\ \frac{d(w1)}{d(w)} d_i(w) & \text{if } d(w) \neq 0 \\ 0 & \text{otherwise,} \end{cases} \\ d_i(w0) &= 2^s d_i(w) - d_i(w1), \end{aligned}$$

where $d(\lambda) = 1$. Let $d' = \sum_{i=0}^\infty 2^{-i} d_i$. Then d' is a polynomial-time computable s-gale. Let $A \in \mathcal{C}$ and $B = L(M_i) \in \mathrm{DTIME}(2^{cn})$. Then for all $n \in \mathbb{N}$, $d_i((A \cup B)[0..n-1]) \geq d(A[0..n-1])$. Because $A \in S^\infty[d]$, $A \cup B \in S^\infty[d_i] \subseteq S^\infty[d']$. $\square$

4 Dimension of $\mathrm{P_m}(\mathrm{DENSE}^c)$

Lutz and Mayordomo [9] proved that a superclass of $\mathrm{P_m}(\mathrm{DENSE}^c)$ has p-measure 0, so $\mu_{\mathrm{p}}(\mathrm{P_m}(\mathrm{DENSE}^c)) = 0$. In this section we prove the stronger result that $\dim_{\mathrm{p}}(\mathrm{P_m}(\mathrm{DENSE}^c)) = 0$.

We use the binary entropy function $\mathcal{H} : [0,1] \to [0,1]$ defined by

$$\mathcal{H}(x) = \begin{cases} -x \log x - (1-x)\log(1-x) & \text{if } x \in (0,1) \\ 0 & \text{if } x \in \{0,1\}. \end{cases}$$

Lemma 4.1 *For all $n \in \mathbb{N}$ and $0 \le k \le n$,*

$$\binom{n}{k} \le \frac{n^n}{k^k (n-k)^{(n-k)}} = 2^{\mathcal{H}(\frac{k}{n})n}.$$

Lemma 4.1 appears as an exercise in [1]. The following lemma is also easy to verify.

Lemma 4.2 *For all $\epsilon \in (0,1)$,*

$$\mathcal{H}(2^{(n^\epsilon - n)})2^n = o(2^{\epsilon n}).$$

We now show that only a p-dimension 0 set of languages are $\le^{\mathrm{P}}_{\mathrm{m}}$-reducible to non-dense languages.

Theorem 4.3

$$\dim_{\mathrm{p}}(\mathrm{P_m}(\mathrm{DENSE}^c)) = 0.$$

Proof: Let $s > 0$ be rational. It suffices to show that $\dim_{\mathrm{p}}(\mathrm{P_m}(\mathrm{DENSE}^c)) \le s$.

Let $\{(f_m, \epsilon_m)\}_{m \in \mathbb{N}}$ be a standard enumeration of all pairs of polynomial-time computable functions $f_m : \{0,1\}^* \to \{0,1\}^*$ and rationals $\epsilon_m \in (0,1)$. Define

$$A_{m,n} = \left\{ u \in \{0,1\}^{2^{n+1}-1} \;\middle|\; \begin{array}{l} (\forall i,j \ge 2^{\frac{n}{2}})(f_m(s_i) = f_m(s_j) \Rightarrow u[i] = u[j]) \\ \text{and } |\{f_m(s_i) | i \ge 2^{\frac{n}{2}} \text{ and } u[i] = 1\}| \le 2^{n^{\epsilon_m}} \end{array} \right\}.$$

For each string u with $2^{\frac{n}{2}} \le |u| \le 2^{n+1} - 1$, define the integers

$$\begin{aligned} \mathrm{collision}_{m,n}(u) &= \left| \{(i,j) | 2^{\frac{n}{2}} \le i < j < |u|, f_m(s_i) = f_m(s_j), \text{ and } u[i] \ne u[j]\} \right|, \\ \mathrm{committed}_{m,n}(u) &= \left| \{f_m(s_i) | 2^{\frac{n}{2}} \le i < |u| \text{ and } u[i] = 1\} \right|, \text{ and} \\ \mathrm{free}_{m,n}(u) &= \left| \{f_m(s_i) |\; |u| \le i < 2^{n+1} - 1\} - \{f_m(s_i) | 2^{\frac{n}{2}} \le i < |u|\} \right|. \end{aligned}$$

Then for each u with $|u| \ge 2^{\frac{n}{2}}$ there are

$$\mathrm{count}_{m,n}(u) = \begin{cases} \displaystyle\sum_{i=0}^{2^{n^{\epsilon_m}} - \mathrm{committed}_{m,n}(u)} \binom{\mathrm{free}_{m,n}(u)}{i} & \text{if } \mathrm{collision}_{m,n}(u) = 0 \\ 0 & \text{otherwise} \end{cases}$$

strings v for which $uv \in A_{m,n}$.

Define for each $m, n \in \mathbb{N}$ a function $d_{m,n} : \{0,1\}^* \to [0,\infty)$ by

$$d_{m,n}(u) = \begin{cases} 2^{(s-1)|u|} & \text{if } |u| < 2^{\frac{n}{2}} \\ \frac{\text{count}_{m,n}(u)}{\text{count}_{m,n}(u[0..2^{\frac{n}{2}}-1])} 2^{s|u|-2^{\frac{n}{2}}} & \text{if } 2^{\frac{n}{2}} \leq |u| \leq 2^{n+1}-1 \\ 2^{(s-1)(|u|-2^{n+1}+1)} d(u[0..2^{n+1}-2]) & \text{otherwise.} \end{cases}$$

Then each $d_{m,n}$ is a well-defined s-gale because $\text{count}_{m,n}(u) = \text{count}_{m,n}(u0) + \text{count}_{m,n}(u1)$ for all u. Define a polynomial-time computable s-gale

$$d = \sum_{m=0}^{\infty} 2^{-m} \sum_{n=0}^{\infty} 2^{-n} d_{m,n}.$$

Let $A \leq_{\text{m}}^{\text{P}} D \in \text{DENSE}^c$ by a reduction f running in time n^l. Let ϵ be a positive rational such that for infinitely many n, $|D_{\leq n^l}| < 2^{n^\epsilon}$. Let $m \in \mathbb{N}$ be such that $f_m = f$ and $\epsilon_m = \epsilon$. Using Lemmas 4.1 and 4.2, for each $u \in \{0,1\}^{2^{\frac{n}{2}}}$, we have

$$\begin{aligned} \text{count}_{m,n}(u) &\leq \sum_{i=0}^{2^{n^\epsilon}} \binom{|f(\{0,1\}^{\leq n})|}{i} \\ &\leq (2^{n^\epsilon}+1)\binom{2^{n+1}-1}{2^{n^\epsilon}} \\ &\leq (2^{n^\epsilon}+1) 2^{\mathcal{H}(2^{n^\epsilon - n})2^n} \\ &\leq 2^{2^{\epsilon n}} \\ &\leq 2^{s2^n - 2^{\frac{n}{2}} - 2n} \end{aligned}$$

for all sufficiently large n. Whenever $|D_{\leq n^l}| < 2^{n^\epsilon}$, we have $A[0..2^{n+1}-2] \in A_{m,n}$. Therefore for infinitely many n,

$$\begin{aligned} d(A[0..2^{n+1}-2]) &\geq 2^{-(m+n)} d_{m,n}(A[0..2^{n+1}-2]) \\ &= 2^{-(m+n)} \frac{\text{count}_{m,n}(A[0..2^{n+1}-2])}{\text{count}_{m,n}(A[0..2^{\frac{n}{2}}-1])} 2^{s(2^{n+1}-1)-2^{\frac{n}{2}}} \\ &\geq 2^{-(m+n)} \frac{2^{s(2^{n+1}-1)-2^{\frac{n}{2}}}}{2^{s2^n-2^{\frac{n}{2}}-2n}} \\ &\geq 2^{n-m}. \end{aligned}$$

Therefore $A \in S^\infty[d]$. This shows that $\text{P}_{\text{m}}(\text{DENSE}^c) \subseteq S^\infty[d]$, from which it follows that $\dim_{\text{p}}(\text{P}_{\text{m}}(\text{DENSE}^c)) = 0$. □

5 Main Theorem

Theorem 5.1 *If $\dim_{\text{p}}(\text{NP}) > 0$, then for all $\epsilon > 0$ there exists a $\delta > 0$ such that any 2^{n^δ}-time approximation algorithm for* MAX3SAT *has performance ratio less than $\frac{7}{8} + \epsilon$ on a dense set of satisfiable instances.*

Proof: We prove the contrapositive. Let $\epsilon > 0$ be rational. For any MAX3SAT approximation algorithm $\mathcal{A}$, define the set

$$F_{\mathcal{A}} = \left\{ x \in \text{3SAT} \,\middle|\, \mathcal{A}(x) < \frac{7}{8} + \epsilon \right\}.$$

Assume that for each $\delta > 0$, there exists a 2^{n^δ}-time approximation algorithm $\mathcal{A}_\delta$ for MAX3SAT with $F_{\mathcal{A}_\delta} \in \mathrm{DENSE}^c$. By Theorem 4.3 and Lemma 3.4, it is sufficient to show that $\mathrm{NP} \subseteq \mathrm{P_m}(\mathrm{DENSE}^c) \uplus \mathrm{DTIME}(2^n)$.

Let $B \in \mathrm{NP}$ and let r be a $\leq_\mathrm{m}^\mathrm{P}$-reduction of B to SAT. Let n^k be an almost-everywhere time bound for computing $f_\epsilon \circ r$ where f_ϵ is as in Theorem 2.1. Then

$$\begin{aligned} x \in B &\iff r(x) \in \mathrm{SAT} \\ &\iff \mathrm{MAX3SAT}((f_\epsilon \circ r)(x)) = 1 \\ &\iff \mathcal{A}_{\frac{1}{k}}((f_\epsilon \circ r)(x)) \geq \tfrac{7}{8} + \epsilon \text{ or } (f_\epsilon \circ r)(x) \in F_{\mathcal{A}_{\frac{1}{k}}}. \end{aligned}$$

Define the languages

$$C = \left\{ x \,\middle|\, (f_\epsilon \circ r)(x) \in F_{\mathcal{A}_{\frac{1}{k}}} \right\} \text{ and } D = \left\{ x \,\middle|\, \mathcal{A}_{\frac{1}{k}}((f_\epsilon \circ r)(x)) \geq \frac{7}{8} + \epsilon \right\}.$$

Then $B = C \cup D$, $C \leq_\mathrm{m}^\mathrm{P} F_{\mathcal{A}_{\frac{1}{k}}} \in \mathrm{DENSE}^c$, and D can be decided in time $2^{(n^k)^{\frac{1}{k}}} = 2^n$ for all sufficiently large n, so $B \in \mathrm{P_m}(\mathrm{DENSE}^c) \uplus \mathrm{DTIME}(2^n)$. □

Theorem 5.1 provides a strong positive answer to Problem 8 of Lutz and Mayordomo [8]:

> Does $\mu_\mathrm{p}(\mathrm{NP}) \neq 0$ imply an exponential lower bound on approximation schemes for MAXSAT?

We observe that a weaker positive answer can be more easily obtained by using a simplified version of our argument to prove the following result.

Proposition 5.2 *If*

$$\mathrm{NP} \not\subseteq \bigcap_{\alpha > 0} \mathrm{DTIME}\left(2^{n^\alpha}\right),$$

then for all $\epsilon > 0$ there exists a $\delta > 0$ such that there does not exist a 2^{n^δ}-time $(\frac{7}{8}+\epsilon)$-approximation algorithm for MAX3SAT.

6 Conclusion

We close by summarizing the inapproximability results for MAX3SAT derivable from various strong hypotheses in the following figure.

$\boxed{\mu_{\mathrm{p}}(\mathrm{NP}) \neq 0}$		
$\Downarrow$		
$\boxed{\dim_{\mathrm{p}}(\mathrm{NP}) > 0}$	$\Rightarrow$	There exists a $\delta > 0$ such that any 2^{n^δ}-time approximation algorithm for MAX3SAT has performance ratio less than $\frac{7}{8} + \epsilon$ on a dense set of satisfiable instances.
$\Downarrow$		$\Downarrow$
$\boxed{\mathrm{NP} \not\subseteq \bigcap_{\alpha>0} \mathrm{DTIME}\left(2^{n^\alpha}\right)}$	$\Rightarrow$	There exists a $\delta > 0$ such that no 2^{n^δ}-time $\left(\frac{7}{8} + \epsilon\right)$-approximation algorithm for MAX3SAT exists.
$\Downarrow$		$\Downarrow$
$\boxed{\mathrm{P} \neq \mathrm{NP}}$	$\Rightarrow$	No polynomial-time $\left(\frac{7}{8} + \epsilon\right)$-approximation algorithm for MAX3SAT exists.

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References

[1] T.H. Cormen, C.E. Leiserson, and R. L. Rivest. *Introduction to Algorithms.* MIT Press/McGraw-Hill, Cambridge, Massachusetts, 1990.

[2] E. Dantsin, M. Gavrilovich, E.A. Hirsch, and B. Konev. MAX SAT approximation beyond the limits of polynomial-time approximation. *Annals of Pure and Applied Logic*. To appear.

[3] E. Halperin and U. Zwick. Approximation algorithms for MAX 4-SAT and rounding procedures for semidefinite programs. In *IPCO: 7th Integer Programming and Combinatorial Optimization Conference*, 1999.

[4] J. Håstad. Some optimal inapproximability results. In *Proceedings of the Twenty-Ninth Annual ACM Symposium on Theory of Computing*, pages 1–10, 1997.

[5] H. Karloff and U. Zwick. A 7/8-approximation algorithm for MAX 3SAT? In *38th Annual Symposium on Foundations of Computer Science*, pages 406–415, 1997.

[6] J. H. Lutz. Almost everywhere high nonuniform complexity. *Journal of Computer and System Sciences*, 44:220–258, 1992.

[7] J. H. Lutz. Dimension in complexity classes. In *Proceedings of the Fifteenth Annual IEEE Conference on Computational Complexity*, pages 158–169. IEEE Computer Society Press, 2000.

[8] J. H. Lutz and E. Mayordomo. Twelve problems in resource-bounded measure. *Bulletin of the European Association for Theoretical Computer Science*, 68:64–80, 1999.

[9] J.H. Lutz and E. Mayordomo. Measure, stochasticity, and the density of hard languages. *SIAM Journal on Computing*, 23(4):762–779, 1994.