Enumerating Rationals Without Repetitions

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April 8, 2009

Abstract

Cantor showed that the positive rational numbers ($\mathbb{Q}^+$) are countable by his well known diagonal enumeration of the fractions. In that enumeration, fractional representatives for each positive rational number are encountered infinitely often. It is an interesting problem to enumerate the positive rationals without repetitions, *i.e.* once an fraction from the equivalence class of a rational has been enumerated, to never enumerate another member of the equivalence class. More formally, an *enumeration* of a set S is a function $f : \mathbb{N} \to S$ that is onto. An *enumeration without repetitions* is an enumeration that is both one-to-one and onto. The Calkin-Wilf tree is an infinite binary tree of reduced fractions containing one representative for each positive rational number. We present an enumeration f for $\mathbb{Q}^+$ based on an efficient unfolding of a path in the Calkin-Wilf tree to the k^{th} rational. We prove that f is an enumeration without repetitions by exhibiting a function $g : \mathbb{Q}^+ \to \mathbb{N}$ and showing that g is the inverse of f thereby proving f is one-to-one and onto. This presentation has been formalized in ACL2.