Proof extraction from multi-succedent intuitionistic derivations

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5. Translation from **G3im** to **G3i**
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Proof Search in Type Theory (NUPRL)

Proof search in the NUPRL proof assistant is based on an automated theorem prover, using (in effect) a multi-succedent intuitionistic calculus; from this algorithms (i.e. ordinary lambda terms) are extracted using a translation.

Multi-succedent calculus is preferred for various reasons: use of efficient matrix/connection methods (Bibel, Wallen) and label unification (Otten); complexity issue (see below); perhaps historical reasons. From the point of view of provability, all these calculi (to follow) are equivalent. From the perspective of finding algorithms, whether they are or are not is unclear.

Gentzen/Kleene calculus G3i[p]

$$\frac{}{\Gamma, \bot \Rightarrow C}\ L\bot \qquad \frac{}{\Gamma, A \Rightarrow A}\ Ax$$

$$\frac{\Gamma, A \supset B \Rightarrow A \quad \Gamma, A \supset B, B \Rightarrow C}{\Gamma, A \supset B \Rightarrow C}\ L\supset \qquad \frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \supset B}\ R\supset$$

$$\frac{\Gamma, A \vee B, A \Rightarrow C \quad \Gamma, A \vee B, B \Rightarrow C}{\Gamma, A \vee B \Rightarrow C}\ L\vee \qquad \frac{\Gamma \Rightarrow A_i}{\Gamma \Rightarrow A_1 \vee A_2}\ R\vee_i$$

$$\frac{\Gamma, A_1 \wedge A_2, A_i \Rightarrow C}{\Gamma, A_1 \wedge A_2 \Rightarrow C}\ L\wedge_i \qquad \frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B}\ R\wedge$$

Gentzen/Maehara/Kleene calculus G3im[p]

$$\frac{}{\Gamma, \bot \Rightarrow \Delta}\, L\bot \qquad \frac{}{\Gamma, A \Rightarrow A, \Delta}\, Ax$$

$$\frac{\Gamma, A \supset B \Rightarrow A, \Delta \quad \Gamma, A \supset B, B \Rightarrow \Delta}{\Gamma, A \supset B \Rightarrow \Delta}\, L\supset \qquad \frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \supset B, \Delta}\, R\supset$$

$$\frac{\Gamma, A \vee B, A \Rightarrow \Delta \quad \Gamma, A \vee B, B \Rightarrow \Delta}{\Gamma, A \vee B \Rightarrow \Delta}\, L\vee \qquad \frac{\Gamma \Rightarrow A, B, \Delta}{\Gamma \Rightarrow A \vee B, \Delta}\, R\vee$$

$$\frac{\Gamma, A_1 \wedge A_2, A_i \Rightarrow \Delta}{\Gamma, A_1 \wedge A_2 \Rightarrow \Delta}\, L\wedge_i \qquad \frac{\Gamma \Rightarrow A, \Delta \quad \Gamma \Rightarrow B, \Delta}{\Gamma \Rightarrow A \wedge B, \Delta}\, R\wedge$$

Complexity (Egly-Schmitt)

Egly and Schmitt [1] showed that there is a sequence (S_n) of intuitionistic sequents s.t., for each n, S_n is derivable in **G3im** (using a derivation with $6n-2$ leaves) but that every derivation of S_n in **G3i** has at least 2^n leaves. Thus, **G3i** cannot polynomially simulate **G3im**. (The converse is known, that **G3im** can polynomially simulate **G3i**.)

Translation from G3i into G3im

Straightforward use of Weakening on the right, e.g. a step

$$\dfrac{\dfrac{\dots}{\Gamma \Rightarrow A}}{\Gamma \Rightarrow A \vee B}$$

becomes

$$\dfrac{\dfrac{\dfrac{\dots}{\Gamma \Rightarrow A}}{\Gamma \Rightarrow A, B}\, RW}{\Gamma \Rightarrow A \vee B}$$

from which uses of RW can be eliminated (by pushing it up to leaves or to $R\supset$ and $R\forall$ inferences). This elimination adds formulae to the RHS, hence the multiple succedents.

Translation from G3im into G3i

1. Via G3i+CUT (e.g. Cuts distribute & over $\vee$)

2. Schmitt & Kreitz (use of modified **G3i**, with more complex initial sequents)

3. Mints (permutation argument)

4. Egly & Schmitt (another permutation argument)

The last three are complex, hard to analyse w.r.t. relationship with natural deduction proofs. We therefore consider only the first.

Translation from G3im to G3i + CUT

Wherever Δ is a multiset, let δ be a disjunction of all the elements in Δ. It is then routine to show that if $\Gamma \Rightarrow \Delta$ is derivable in **G3im** then $\Gamma \Rightarrow \delta$ is derivable in **G3i**. Suppose for example the final step is by $R\wedge$, giving $\Gamma \Rightarrow A \wedge B, \Delta'$. A $R\wedge$ step on the transforms of the derivations of the premisses followed by a cut with $(A \vee \delta') \wedge (B \vee \delta') \Rightarrow (A \wedge B) \vee \delta'$ gives us $\Gamma \Rightarrow (A \wedge B) \vee \delta'$.

There are alas tedious issues concerning the order in which the elements of a multiset are disjoined. Some easy simplifications are possible (e.g. doing two cuts before rather than one step after the $R\wedge$ step).

Cut-elimination in G3i+CUT

We now have, starting from a **G3im** derivation, a derivation in **G3i+CUT**. Our goal is to get a natural deduction (represented as a lambda term). Use of a lambda notation for derivations is, once we are in **G3i+CUT**, routine. We can now go by several routes: translate directly to ordinary lambda terms and normalise, or eliminate cuts and translate to normal lambda terms.

We chose (for reasons now obscure) the second route, although there are *caveats* (see e.g. [2,3]):

1. Not all cut elimination systems in **G3i+CUT** are SN;

2. Not all cut elimination systems in **G3i+CUT** are confluent;

3. Not all cut elimination systems in **G3i+CUT** simulate beta-reduction.

Cut-elimination in G3i+CUT, 2

[We chose the second route], using a particular SN system of cut reduction rules. The lack of confluence shouldn't be a problem, provided that the possible cut-free terms one obtains all have the same meaning as (normal) lambda terms; likewise, the failure to simulate beta-reduction isn't a problem, we are only interested in the end result, a normal lambda term.

In part, checking the first of these provisos means checking, in lambda calculus, lots of equations like

$$[w/x][xM/y]N = [w[w/x]M/y][w/x]N$$

corresponding to cut-reduction steps.

Conjecture and issues

We *conjecture* that, whenever a **G3i** derivation d is found for a sequent, its interpretation as a normal lambda term can also be obtained by interpreting (as above) at least one **G3im** derivation, namely the interpretation of d in **G3im**. Thus, that the interpretation (using cuts and cut-elimination) from **G3im** into the space of normal lambda terms is surjective.

We are part-way to establishing this. Whether this is true for the translations of Schmitt-Kreitz, Mints or Egly-Schmitt is, we guess, much harder to establish (except perhaps negatively).

Whether the interpretation is surjective if we consider only small **G3im** derivations is unclear; a complexity argument might decide it. By "small" is meant, for example, linear in the size of the end-sequent, thus allowing the small **G3im** derivations of the Egly-Schmitt examples.

Experiments

Caldwell has implemented the above methods, and experimented; the results are not yet conclusive.

Of course, the correctness of the implementation also has to be verified. . . .

Comment

Whether **G3im** is appropriate for searching for algorithms is not clear; one of its advantages (if used root-first as a sequent calculus) is the invertibility of almost all rules, but then one immediately ignores lots of possible algorithms. That seems to be a separate issue. In any case, a Herbelin-style calculus (as in Lengrand's talk) seems more appropriate.

References

1. Egly, Schmitt. *On Intuitionistic Proof Transformations . . .*, Fund. Inform. 1999.

2. Kikuchi. *On a Local-Step Cut-Elimination Procedure for the Intuitionistic Sequent Calculus*, LPAR 2006.

3. Kikuchi. *Confluence of Cut-Elimination Procedures for the Intuitionistic Sequent Calculus*, CiE 2007.

4. Maehara. *Eine Darstellung der intuitionistiche Logik in der klassischen*, Nagoya Math J. 1954.

5. Mints. *A Short Introduction to Intuitionistic Logic*, Kluwer 2000.

6. Schmitt, Kreitz. *On transforming intuitionistic matrix proofs into standard sequent proofs*, TABLEAUX workshop, 1995.