SIGACT News Complexity Theory Column 48

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Introduction to Complexity Theory Column 48

Here is a real gift to the field from David Johnson: After a thirteen year intermission, David is restarting his NP-completeness column. His column will now appear about twice yearly in *ACM Transactions on Algorithms*. Welcome back David, and thanks! And for those for whom a diet of two per year won't do, meals past can be found at `http://www.research.att.com/~dsj/columns.html`.

As to the Complexity Theory Column, warmest thanks to Elvira, Jack, and John for their wonderful guest column on *The Fractal Geometry of Complexity Classes* in this issue. Upcoming articles include Neil Immerman on *Recent Progress in Descriptive Complexity*, Piotr Faliszewski and me on *Open Questions in the Theory of Semifeasible Computation*, and Omer Reingold on a topic TBA.

Guest Column: The Fractal Geometry of Complexity Classes[1]

***John M. Hitchcock*[2]** ***Jack H. Lutz*[3]** ***Elvira Mayordomo*[4]**

1 Introduction

Research developments since early 2000 have transformed powerful methods from geometric measure theory into high-precision quantitative tools for investigating the structure of complexity classes. This column gives an overview of the very early days (i.e., all the days so far) of this line of inquiry, along with a little bit of its prehistory.

We begin our story not at the beginning (Euclid), but in 1918, when the mathematician Felix Hausdorff [22] showed how to assign each subset X of an arbitrary metric space a dimension that is now called the *Hausdorff dimension* of X, denoted by $\dim_{\mathrm{H}}(X)$. For "smooth" sets X, Hausdorff dimension agrees with our most basic intuitions (e.g., smooth curves are 1-dimensional; smooth surfaces are 2-dimensional), but Hausdorff noted at the outset that many sets X have

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"fractional dimension," by which he meant that $\dim_{\mathrm{H}}(X)$ may be any nonnegative real number, not necessarily an integer. Hausdorff dimension has become one of the most powerful tools of *fractal geometry*, an extensively developed subfield of geometric measure theory with applications throughout the sciences [17, 63, 14, 16, 15, 12, 13, 57]. In the 1980s Tricot [75] and Sullivan [74] independently developed a dual of Hausdorff dimension called *packing dimension* and denoted by $\dim_{\mathrm{P}}(X)$, that now rivals Hausdorff dimension's importance in such investigations. In general, $\dim_{\mathrm{H}}(X) \leq \dim_{\mathrm{P}}(X)$, with equality if X is sufficiently "regular."

The connection between these fractal dimensions and computational complexity was made via *martingales*. A martingale is a strategy for betting on the successive bits of infinite binary sequences with fair payoffs. Martingales were introduced by Ville [77] in 1939 (having been implicit in earlier works of Lévy [42, 43]). In the early 1970s, following Martin-Löf's use of constructive measure theory to give the first satisfactory definition of the randomness of individual infinite binary sequences [56], Schnorr [66, 67, 68, 69] used martingales extensively in his investigations of randomness.[5] Of some relevance to our story here, Schnorr used the *growth rates* of computable martingales (rates at which their capital grows as they bet on various sequences) to investigate the relationships among various randomness notions.

The first observations connecting growth rates of martingales to fractal dimensions were made by Ryabko [64, 65] and Staiger [72] in the 1990s. These observations were quantitative theorems relating the Hausdorff dimension of a set X of binary sequences to the growth rates achievable by computable martingales betting on the sequences in X.[6]

The precise and general nature of the relationship suggested by Ryabko and Staiger's theorems became clear in 2000, when Lutz proved the *gale characterization of Hausdorff dimension* [50, 52]. Briefly, if s is a nonnegative real number (most interesting when $0 < s \leq 1$), an *s-gale* is a function d of the form $d(w) = 2^{(s-1)|w|}d'(w)$, where d' is a martingale [51]. A martingale is thus a 1-gale. The gale characterization says that the Hausdorff dimension of a set X of infinite binary sequences is simply the infimum s for which there exists an s-gale d that, when betting on any element of X, wins unbounded money. This is exactly the infimum s such that the above-mentioned martingale d' grows, in an infinitely-often sense, at least as fast as $2^{(1-s)|w|}$ on prefixes w of sequences in X.[7] Thus the gale characterization gives an exact relationship between the Hausdorff dimension of X and the growth rates achievable by (not necessarily computable) martingales betting on the sequences in X. In 2004, Athreya, Lutz, Hitchcock, and Mayordomo [4] proved that packing dimension also admits a gale characterization, the only difference being that the s-gale's money is now required to converge to infinity.

The most important benefit of the gale characterization of fractal dimension is that it enables one to define *effective versions* of fractal dimension by imposing various computability and complexity constraints on the gales. For example, the *feasible dimension* of a set X (written $\dim_{\mathrm{p}}(X)$, the p-*dimension* of X) is the infimum s for which there exists a *polynomial-time computable* s-gale d that wins unbounded money on every sequence in X. Other effective dimensions that are useful in complexity theory include p_2-*dimension* (quasipolynomial-time dimension) and pspace-*dimension*. Effective dimensions that have been useful *outside* of complexity theory include *constructive dimension*, introduced in [51, 53] and surveyed in the forthcoming book by Downey and Hirschfeldt

[5]Doob [10] modified Ville's definition, and the resulting notion is now a central notion of probability theory. Schnorr's investigations of randomness, Lutz's development of resource-bounded measure [47, 49], and the effective fractal dimensions discussed in this column all use Ville's definition rather than Doob's. The necessity of this choice is explained in [30].

[6]See pages 72-73 of [53] for a more detailed review of the above-mentioned work of Schnorr, Ryabko, and Staiger, including work by the latter two and Cai and Hartmanis [8] relating Kolmogorov complexity to Hausdorff dimension.

[7]By this we mean that each sequence in X has infinitely many prefixes w at which $d'(w) \geq 2^{(1-s)|w|}$.

[11]; *computable dimension*, introduced in [52] and discussed in [11]; and *finite-state dimension* [9].

Our objective here is to survey applications of effective dimension in computational complexity. We do this in two stages. In Part I (Sections 2-7) we discuss applications of feasible dimension and other resource-bounded dimensions to questions about circuit-size complexity, polynomial-time degrees, the dimension of NP, zero-one laws, and oracle classes. Part II (Sections 8-11) concerns *scaled dimensions*. As their name suggests, these are versions of resource-bounded dimension that have been "rescaled" to better fit the phenomena that they are measuring. Applications of scaled dimensions to circuit-size complexity, polynomial-time degrees, and the dimension of NP are discussed.

Both parts of this survey begin by introducing the fundamental properties of the fractal dimensions to be discussed.

I. Dimension

2 Foundations

Formally, if $s \in [0, \infty)$, then an *s-gale* is a function $d : \{0,1\}^* \to [0, \infty)$ satisfying the condition

$$d(w) = 2^{-s}[d(w0) + d(w1)] \tag{2.1}$$

for all $w \in \{0,1\}^*$ [52]. A *martingale* is a 1-gale.

Intuitively, we think of a gale d as a strategy for betting on the successive bits of a sequence S. (All *sequences* here are infinite and binary.) The quantity $d(w)$ is interpreted as the capital (amount of money) that a gambler using the strategy d has after betting on the successive bits of the prefix w of S. The parameter s regulates the fairness of the payoffs via identity (2.1). If $s = 1$, the payoffs are fair in the usual sense that the conditional expectation of the gambler's capital $d(wb)$, given that w has occurred, is precisely $d(w)$, the gambler's actual capital before betting on the last bit of wb. If $s < 1$, then the payoffs are unfair, and the smaller s is, the more unfair the payoffs are.

A gale d *succeeds* on a sequence S if

$$\limsup_{w \to S} d(w) = \infty$$

and *succeeds strongly* on S if

$$\liminf_{w \to S} d(w) = \infty.$$

The *success set* $S^\infty[d]$ of a gale d is the set of all sequences on which d succeeds. The *strong success set* $S^\infty_{\mathrm{str}}[d]$ is the set of all sequences on which d succeeds strongly.

The original definitions of Hausdorff dimension [22] and packing dimension [75, 74] are ingenious, but the following characterizations are usually more convenient for work in the theory of computing.

Theorem 2.1. (gale characterization of fractal dimension) *Let X be a set of sequences.*

1. (Lutz [52]) $\quad \dim_{\mathrm{H}}(X) = \inf\{s \mid$ *there is an s-gale d such that* $X \subseteq S^\infty[d]\}$.
2. (Athreya et al. [4]) $\quad \dim_{\mathrm{P}}(X) = \inf\{s \mid$ *there is an s-gale d such that* $X \subseteq S^\infty_{\mathrm{str}}[d]\}$.

Intuitively, Theorem 2.1 says that the fractal dimension of a set X of sequences is the *most hostile environment* (i.e., most unfair payoff parameter s) in which a gambler can win on every sequence in X. Of course, the word "win" here means "succeed" in the case of Hausdorff dimension and "succeed strongly" in the case of packing dimension.

It is easy to see that $0 \leq \dim_{\mathrm{H}}(X) \leq \dim_{\mathrm{P}}(X) \leq 1$ holds in any case. Both of these fractal dimensions are *monotone* (i.e., $X \subseteq Y$ implies $\dim(X) \leq \dim(Y)$), *countably stable* (i.e., $\dim(\bigcup_{i=0}^{\infty} X_i) = \sup_i \dim(X_i)$), and *nonatomic* (i.e., $\dim(\{S\}) = 0$ for each sequence S) [16]. In particular, every countable set of sequences has Hausdorff and packing dimension 0.

We say that a gale $d : \{0,1\}^* \to [0,\infty)$ is p-*computable* if there is a function $\hat{d} : \{0,1\}^* \times \mathbb{N} \to \mathbb{Q}$ such that $\hat{d}(w,r)$ is computable in time polynomial in $|w| + r$ and $|\hat{d}(w,r) - d(w)| \leq 2^{-r}$ holds for all w and r. Gales that are pspace-computable, p_2-computable, p_3-computable, etc., are defined analogously, with $\hat{d}(w,r)$ required to be computable in space polynomial in $|w| + r$ (with output *included* as space) in the case of pspace-computability, computable in $2^{(\log(|w|+r))^{O(1)}}$ time in the case of p_2-computability, and computable in $2^{2^{(\log\log(|w|+r))^{O(1)}}}$ time in the case of p_3-computability.

We are finally ready to bring this all home to complexity classes. We identify each language (i.e., decision problem) $A \subseteq \{0,1\}^*$ with its characteristic sequence, whose n^{th} bit is 1 if the n^{th} string in $\{0,1\}^*$ (in the standard enumeration $\lambda, 0, 1, 00, 01, \ldots$) is an element of A, and 0 otherwise. We say that a gale succeeds on A if it succeeds on the characteristic sequence of A and similarly for strong success. We now show how to define fractal dimension in the complexity classes $\mathrm{E} = \mathrm{TIME}(2^{\text{linear}})$, $\mathrm{E}_2 = \mathrm{EXP} = \mathrm{TIME}(2^{\text{polynomial}})$, $\mathrm{E}_3 = \mathrm{TIME}(2^{\text{quasipolynomial}})$, and $\mathrm{ESPACE} = \mathrm{SPACE}(2^{\text{linear}})$.

Definition. ([52, 4]) Let X be a set of languages.

1. If Δ is any of the resource bounds p, p_2, p_3, or pspace, then the Δ-*dimension* of X is

$$\dim_\Delta(X) = \inf\{s \mid \text{there is a } \Delta\text{-computable } s\text{-gale } d \text{ such that } X \subseteq S^\infty[d]\ \},$$

   and the *strong* Δ-*dimension* of X is

$$\mathrm{Dim}_\Delta(X) = \inf\{s \mid \text{there is a } \Delta\text{-computable } s\text{-gale } d \text{ such that } X \subseteq S^\infty_{\text{str}}[d]\}.$$

2. The *dimension of* X *in* E is $\dim(X \mid \mathrm{E}) = \dim_{\mathrm{p}}(X \cap \mathrm{E})$.

3. The *dimension of* X *in* EXP is $\dim(X \mid \mathrm{EXP}) = \dim_{\mathrm{p}_2}(X \cap \mathrm{EXP})$.

4. The *dimension of* X *in* E_3 is $\dim(X \mid \mathrm{E}_3) = \dim_{\mathrm{p}_3}(X \cap \mathrm{E}_3)$.

5. The *dimension of* X *in* ESPACE is $\dim(X \mid \mathrm{ESPACE}) = \dim_{\mathrm{pspace}}(X \cap \mathrm{ESPACE})$.

6. The *strong dimensions* $\mathrm{Dim}(X \mid \mathrm{E})$, $\mathrm{Dim}(X \mid \mathrm{EXP})$, $\mathrm{Dim}(X \mid \mathrm{E}_3)$, and $\mathrm{Dim}(X \mid \mathrm{ESPACE})$ are defined analogously.

By Theorem 2.1, $\dim(X \mid \mathcal{C})$ and $\mathrm{Dim}(X \mid \mathcal{C})$ are analogs of Hausdorff and packing dimension, respectively. It was shown in [52, 4] that these analogs are in fact well-behaved, *internal* dimensions in the classes $\mathcal{C}$ that we have mentioned. In all these classes, $0 \leq \dim(X \mid \mathcal{C}) \leq \mathrm{Dim}(X \mid \mathcal{C}) \leq 1$ hold, with $\dim(\mathcal{C} \mid \mathcal{C}) = 1$.

A crucial property of fractal dimensions in complexity classes is that they are *robust*, meaning that they admit several equivalent formulations. Although space does not permit us to elaborate, we mention two of these. First, Hitchcock [26] proved that, for any of the above-mentioned resource bounds Δ, Δ-dimension is precisely unpredictability by Δ-computable predictors in the log-loss model of prediction, and it was shown in [4] that strong Δ-dimension admits a dual unpredictability characterization. When combined with results by Fortnow and Lutz [19], this characterization also yielded new relationships between log-loss prediction and linear-loss prediction.

Resource-bounded fractal dimensions have also been characterized in terms of data compression. Hitchcock [25, 35] obtained a characterization of pspace-dimension in terms of space-bounded Kolmogorov complexity that is closely analogous to Mayordomo's Kolmogorov complexity characterization of constructive dimension [59]. Obtaining a data compression characterization of p-dimension was more problematic, but López-Valdés and Mayordomo [45] have recently achieved this.

We briefly mention the relationships between resource-bounded dimension and two of its forerunners: resource-bounded measure [47, 49] and resource-bounded category [46]. For measure, the relationship is straightforward: a set X has Δ-measure 0 if there is a Δ-computable martingale (i.e., a 1-gale) that succeeds on all of its elements, so $\dim(X \mid \mathcal{C}) < 1$ implies $\mu(X \mid \mathcal{C}) = 0$ in any case, but the converse has many counterexamples. For resource-bounded category, the situation is different for dimension and strong dimension. It is easy to see that a set X can be meager in $\mathcal{C}$ while satisfying either $\dim(X \mid \mathcal{C}) = 0$ or $\dim(X \mid \mathcal{C}) = 1$ and that X can also be comeager in $\mathcal{C}$ while satisfying either of these conditions, so resource-bounded dimension is independent of resource-bounded category. On the other hand, Hitchcock and Pavan [33] have recently shown that $\mathrm{Dim}(X \mid \mathcal{C}) < 1$ implies that X is meager in $\mathcal{C}$, i.e., that there is a definite relationship between resource-bounded strong dimension and resource-bounded category.

3 Circuit-Size Complexity I

The relationship between uniform and nonuniform complexity measures is one of the most important issues in computational complexity. One approach to this issue is to investigate the sizes of (nonuniform) circuit-size complexity classes in time- and space-complexity classes. Lutz initiated this approach in [47] by showing that the Boolean circuit-size complexity class $\mathrm{SIZE}^{\mathrm{i.o.}}(\frac{2^n}{n})$ has measure 0 in ESPACE, thereby improving Shannon's [71] lower bound of $\alpha\frac{2^n}{n}$ for every $\alpha < 1$ on almost every language.

In this section we use fractal dimensions in complexity classes to obtain more quantitative results along these lines. To make our notation precise, the *circuit-size complexity* of a language $A \subseteq \{0,1\}^*$ is the function $\mathrm{CS}_A : \mathbb{N} \to \mathbb{N}$, where $\mathrm{CS}_A(n)$ is the number of gates in the smallest n-input Boolean circuit that decides $A \cap \{0,1\}^n$. For each function $f : \mathbb{N} \to \mathbb{N}$, we define the circuit-size complexity classes

$$\mathrm{SIZE}(f) = \{A \subseteq \{0,1\}^* \mid (\forall^\infty n)\mathrm{CS}_A(n) \leq f(n)\},$$

$$\mathrm{SIZE}^{\mathrm{i.o.}}(f) = \{A \subseteq \{0,1\}^* \mid (\exists^\infty n)\mathrm{CS}_A(n) \leq f(n)\}.$$

Then $\mathrm{P/poly} = \bigcup_{k\in\mathbb{N}} \mathrm{SIZE}(n^k)$ is the class of languages decidable by polynomial-size circuits, and similarly $\mathrm{P/poly}^{\mathrm{i.o.}} = \bigcup_{k\in\mathbb{N}} \mathrm{SIZE}^{\mathrm{i.o.}}(n^k)$.

The following theorem shows how the dimension of a circuit-size complexity class varies with the bound that defines it.

Theorem 3.1. ([52, 4]) *For every $\alpha \in [0,1]$,*

$$\dim\left(\mathrm{SIZE}\left(\alpha\tfrac{2^n}{n}\right)\middle|\,\mathrm{ESPACE}\right) = \mathrm{Dim}\left(\mathrm{SIZE}\left(\alpha\tfrac{2^n}{n}\right)\middle|\,\mathrm{ESPACE}\right) = \alpha.$$

For time-complexity classes, the main uniform versus nonuniform complexity separations are open. For example, it is not known whether EXP is contained in P/poly. Lutz [47] showed that $\mathrm{P/poly}^{\mathrm{i.o.}}$ has measure 0 in the larger class E_3, and that for each fixed $k \geq 1$, $\mathrm{SIZE}^{\mathrm{i.o.}}(n^k)$ has measure 0 in EXP. This result was improved to dimension 0, but only for the almost-everywhere versions of these classes.

Theorem 3.2. (Hitchcock and Vinodchandran [35]) P/poly *has dimension 0 in* E_3. *For fixed* $k \geq 1$, $\mathrm{SIZE}(n^k)$ *has dimension 0 in* EXP.

More recently, Gu used relationships between Kolmogorov complexity and circuit-size complexity [1] to handle the infinitely-often classes, and to consider strong dimension.

Theorem 3.3. (Gu [21])

1. $\mathrm{Dim}(\mathrm{P/poly} \mid \mathrm{E}_3) = 0$. *For fixed* $k \geq 1$, $\mathrm{Dim}(\mathrm{SIZE}(n^k) \mid \mathrm{EXP}) = 0$.
2. $\dim(\mathrm{P/poly}^{\mathrm{i.o.}} \mid \mathrm{E}_3) = 1/2$. *For fixed* $k \geq 1$, $\dim(\mathrm{SIZE}^{\mathrm{i.o.}}(n^k) \mid \mathrm{EXP}) = 1/2$.
3. $\mathrm{Dim}(\mathrm{P/poly}^{\mathrm{i.o.}} \mid \mathrm{EXP}) = \mathrm{Dim}(\mathrm{P/poly}^{\mathrm{i.o.}} \mid \mathrm{E}_3) = 1$.

Part of the proof of Theorem 3.3 is that *any* infinitely-often defined class must have dimension at least 1/2 and strong dimension 1.

Mayordomo [58] used Stockmeyer's approximate counting of #P functions [73] to show that P/poly has measure 0 in $\Delta_3^{\mathrm{E}} = \mathrm{E}^{\Sigma_2^{\mathrm{P}}}$. The following result used Köbler, Schöning, and Torán's approximate counting of SpanP functions [41] to substantially sharpen this technique, yielding a simultaneous improvement of Mayordomo's theorem and Theorem 3.1.

Theorem 3.4. (Hitchcock and Vinodchandran [35]) *For all* $\alpha \in [0,1]$, $\dim_{\Delta_3^{\mathrm{P}}}(\mathrm{SIZE}(\alpha\frac{2^n}{n})) = \alpha$.

In particular, $\dim(\mathrm{P/poly} \mid \Delta_3^{\mathrm{E}}) = 0$. Another resource-bounded measure result regarding P/poly is a conditional one: Köbler and Lindner [40] showed that if $\mu_{\mathrm{p}}(\mathrm{NP}) \neq 0$, then P/poly has measure 0 in $\mathrm{EXP}^{\mathrm{NP}}$. The full version of [35] uses derandomized approximate counting [70] to improve this to $\dim(\mathrm{P/poly} \mid \mathrm{E}^{\mathrm{NP}}) = 0$ under the same hypothesis.

4 Polynomial-Time Degrees I

In this section we look at degrees and spans of languages under polynomial-time reductions. For a reducibility $\leq_r^{\mathrm{p}}$ and any language A, the $\leq_r^{\mathrm{p}}$-*lower span* of A is the class $\mathrm{P}_r(A)$ of all languages that are $\leq_r^{\mathrm{p}}$-reducible to A; the $\leq_r^{\mathrm{p}}$-*upper span* of A is the class $\mathrm{P}_r^{-1}(A)$ of all languages to which A is $\leq_r^{\mathrm{p}}$-reducible; and the $\leq_r^{\mathrm{p}}$-*degree* of A is the class $\deg_r^{\mathrm{p}}(A) = \mathrm{P}_r(A) \cap \mathrm{P}_r^{-1}(A)$.

Juedes and Lutz [38] proved a *small span theorem* for $\leq_{\mathrm{m}}^{\mathrm{p}}$-reductions in E and EXP. In EXP, this theorem says that, for every $A \in \mathrm{EXP}$, at least one of the lower and upper spans must be small, in the sense that $\mu(\mathrm{P_m}(A) \mid \mathrm{EXP}) = 0$ or $\mu(\mathrm{P_m^{-1}}(A) \mid \mathrm{EXP}) = 0$. Juedes and Lutz also noted that extending this result to $\leq_{\mathrm{T}}^{\mathrm{p}}$-reductions (or even $\leq_{\mathrm{tt}}^{\mathrm{p}}$-reductions [3]) would separate BPP from EXP. Subsequent improvements by Lindner [44], Ambos-Spies, Neis, and Terwijn [3], and Buhrman and van Melkebeek [7] extended the small span theorem to $\leq_{1-\mathrm{tt}}^{\mathrm{p}}$-reductions, $\leq_{k-\mathrm{tt}}^{\mathrm{p}}$-reductions, and $\leq_{n^\alpha-\mathrm{tt}}^{\mathrm{p}}$-reductions ($\alpha < 1$), respectively.

Is there a small span theorem for dimension? That is, must at least one of the lower and upper spans have dimension 0 in E? The first work to consider this yielded negative results. Ambos-Spies, Merkle, Reimann, and Stephan [2] proved that there is no small span theorem for dimension in the case of exponential time. Their main result states that the dimension of the lower span coincides with that of the degree for every language in E.

Theorem 4.1. (Ambos-Spies et al. [2]) *For every* $A \in \mathrm{E}$*,* $\dim(\mathrm{P_m}(A) \mid \mathrm{E}) = \dim(\deg^{\mathrm{p}}_{\mathrm{m}}(A) \mid \mathrm{E})$.

If a set C is $\leq^{\mathrm{p}}_{\mathrm{m}}$-complete for E, then $\mathrm{P_m}(C)$ has dimension 1 in E. Theorem 4.1 says that $\deg^{\mathrm{p}}_{\mathrm{m}}(C)$ also has dimension 1 in E. Since the upper span contains the degree, the upper span also has dimension 1 in E. Therefore the small span theorem fails in a very strong way to extend to dimension.

Theorem 4.1 has some other interesting consequences. For example, if $A \leq^{\mathrm{p}}_{\mathrm{m}} B$, then

$$\dim(\deg^{\mathrm{p}}_{\mathrm{m}}(A) \mid \mathrm{E}) \leq \dim(\deg^{\mathrm{p}}_{\mathrm{m}}(B) \mid \mathrm{E}),$$

even though the degrees are disjoint if $B \not\leq^{\mathrm{p}}_{\mathrm{m}} A$. And for many complexity classes the dimension of the class coincides with the dimension of the class's complete sets. Let NPC denote the NP-complete sets. Then applying the theorem to SAT, we have $\dim(\mathrm{NP} \mid \mathrm{E}) = \dim(\mathrm{NPC} \mid \mathrm{E})$.

We remark that Theorem 4.1 and its corollaries admit extensions in three different ways: to more general polynomial-time reductions (such as $\leq^{\mathrm{p}}_{\mathrm{T}}$) [2], to dimension in larger classes (such as EXP) [28], and to strong dimension [4].

Ambos-Spies et al. also show that the dimension of the many-one degree of a set can take essentially any value (out of a dense subset of (0,1)). In the following, $\mathcal{H}$ is Shannon's binary entropy function $\mathcal{H}(\beta) = -\beta \log \beta - (1-\beta)\log(1-\beta)$.

Theorem 4.2. (Ambos-Spies et al. [2]) *For every computable* $\beta \in [0,1]$ *there is a set* $A \in \mathrm{E}$ *such that* $\dim(\deg^{\mathrm{p}}_{\mathrm{m}}(A) \mid \mathrm{E}) = \mathcal{H}(\beta)$.

Therefore, by Theorem 4.1, the lower spans can take virtually any value. The same proof shows that Theorem 4.2 also holds for strong dimension. In fact, the idea can be extended to show that there are degrees in E with essentially any possible pair of values for the dimension and strong dimension.

Theorem 4.3. (Athreya et al. [4]) *For every pair of computable numbers* $\alpha \leq \beta \in [0,1]$ *there is a set* $A \in \mathrm{E}$ *such that* $\dim(\deg^{\mathrm{p}}_{\mathrm{m}}(A) \mid \mathrm{E}) = \mathcal{H}(\alpha)$ *and* $\mathrm{Dim}(\deg^{\mathrm{p}}_{\mathrm{m}}(A) \mid \mathrm{E}) = \mathcal{H}(\beta)$.

While Theorem 4.1 tells us that there is no small span theorem for dimension, Moser has shown that a small span theorem holds using a mixture of strong dimension and resource-bounded measure, even with reductions that make a fixed number of nonadaptive queries.

Theorem 4.4. (Moser [61]) *For every* $A \in \mathrm{E}$ *and* $k \geq 1$*,* $\mathrm{Dim}(\mathrm{P}_{k-\mathrm{tt}}(A) \mid \mathrm{E}) = 0$ *or* $\mu(\mathrm{P}^{-1}_{k-\mathrm{tt}}(A) \mid \mathrm{E}) = 0$.

5 Zero-One Laws

In the 1990s, van Melkebeek [76] used Impagliazzo and Wigderson's then-new partial derandomization techniques [37] to prove that either BPP has measure 0 in EXP or else BPP = EXP. In his 2004 Ph.D. thesis, Moser strengthened this measure zero-one law to the following dimension zero-one law.

Theorem 5.1. (Moser [60]) $\dim(\mathrm{BPP} \mid \mathrm{EXP}) = 0$ *or* $\mathrm{BPP} = \mathrm{EXP}$.

It is an open question whether zero-one laws hold for the dimensions of ZPP or RP, although measure zero-one laws are known for these classes [76, 36].

6 The Dimension of NP I

What is the dimension of NP in EXP? If $\dim(\mathrm{NP} \mid \mathrm{EXP}) > 0$, then $\mathrm{P} \neq \mathrm{NP}$, since $\dim(\mathrm{P} \mid \mathrm{EXP}) = 0$ [52]. If $\dim(\mathrm{NP} \mid \mathrm{EXP}) < 1$, then $\mathrm{NP} \neq \mathrm{EXP}$, since $\dim(\mathrm{EXP} \mid \mathrm{EXP}) = 1$ [52]. Calculating the dimension of NP in EXP may therefore be difficult. Notwithstanding this, it should be possible to obtain partial results that shed some light on the matter.

The strong hypothesis that $\mu(\mathrm{NP} \mid \mathrm{EXP}) \neq 0$ (i.e., NP does not have measure 0 in EXP) is known to have many plausible consequences that are not known to follow from $\mathrm{P} \neq \mathrm{NP}$ or other qualitative hypotheses [48, 55].

Hitchcock [24] has proven that MAX3SAT is exponentially hard to approximate if $\dim(\mathrm{NP} \mid \mathrm{EXP}) > 0$. This is ostensibly a weaker hypothesis than $\mu(\mathrm{NP} \mid \mathrm{EXP}) \neq 0$ since

$$\mu(\mathrm{NP} \mid \mathrm{EXP}) \neq 0 \Rightarrow \dim(\mathrm{NP} \mid \mathrm{EXP}) = 1 \Rightarrow \dim(\mathrm{NP} \mid \mathrm{EXP}) > 0 \tag{6.1}$$

and neither implication's converse is known to hold.

There is an oracle relative to which the first converse in (6.1) fails to hold. Regarding the second converse, the second author has conjectured that for every $\alpha \in [0, 1]$, there is an oracle A such that $\dim^A(\mathrm{NP}^A \mid \mathrm{EXP}^A) = \alpha$. In contrast, Fortnow [18] has conjectured that we can relativizably prove a zero-one law for NP, stating that $\dim(\mathrm{NP} \mid \mathrm{EXP})$ is either 0 or 1. Any progress on these questions would be interesting.

Most of the aforementioned consequences of $\mu(\mathrm{NP} \mid \mathrm{EXP}) \neq 0$ are not known to follow from a dimension hypothesis on NP. It seems a challenging problem to derive more consequences of $\dim(\mathrm{NP} \mid \mathrm{EXP}) > 0$.

7 Dimensions of Oracle Classes

Bennett and Gill [6] initiated the study of *random oracles* in complexity theory. One of their theorems says that $\mathrm{P}^R \neq \mathrm{NP}^R$ relative to a random oracle R. More precisely, this result says that the set

$$\mathcal{O}_{[\mathrm{P}=\mathrm{NP}]} = \{A \mid \mathrm{P}^A = \mathrm{NP}^A\}$$

of oracles has Lebesgue measure 0. Since then random oracles for complexity theory have been often investigated.

As Hausdorff dimension is capable of distinguishing among the measure 0 sets, it is interesting to consider the dimensions of oracles classes from random oracle results. The following theorem implies that $\mathcal{O}_{[\mathrm{P}=\mathrm{NP}]}$ and many other oracle classes arising in complexity theory all have Hausdorff dimension 1.

Theorem 7.1. (Hitchcock [27]) *If there exists a* paddable *and* relativizable *oracle construction for a complexity-theoretic statement Φ, then the oracle class $\mathcal{O}_{[\Phi]} = \{A \mid \Phi^A$ holds$\}$ has Hausdorff dimension 1.*

It is not known whether the polynomial-time hierarchy is infinite relative to a random oracle, or whether $\mathrm{P} = \mathrm{NP} \cap \mathrm{coNP}$ relative to a random oracle. By the Kolmogorov zero-one law, one of the complementary oracle classes $\mathcal{O}_{[\mathrm{PH\ collapses}]}$ and $\mathcal{O}_{[\mathrm{PH\ is\ infinite}]}$ has measure 1, and the other has measure 0. The same is true for the pair $\mathcal{O}_{[\mathrm{P}=\mathrm{NP}\cap\mathrm{coNP}]}$ and $\mathcal{O}_{[\mathrm{P}\neq\mathrm{NP}\cap\mathrm{coNP}]}$. While we do not know which has measure 1 in either case, it is shown in [27] that all four of these classes have Hausdorff dimension 1.

II. Scaled Dimension

8 Foundations

Scaled dimension [31] was first introduced by the authors of this survey using *scaled gales*, which are generalizations of gales. An equivalent, convenient way to define scaled dimension uses growth rates of martingales.

Recall that $\dim_\Delta(X) \leq s$ if there exists a Δ-computable martingale d such that for all $A \in X$,

$$(\exists^\infty n)\ d(A \upharpoonright n) \geq 2^{n-sn}.$$

Scaled dimension arises using other growth rates in place of $n - sn$. In the general theory, there is a natural hierarchy of growth rates $r_i(s, n)$, one for each integer $i \in \mathbb{Z}$, built around the standard growth rate

$$r_0(s, n) = n - sn.$$

The i^{th} growth rate gives us i^{th}-order scaled dimension. For computational complexity, the most useful orders appear to be $i \in \{-3, -2, 0, 1, 2\}$. (Surprisingly, there has not yet been a compelling application of the -1^{st} order.) In the 1^{st} and 2^{nd} orders of scaled dimension, we use the more rapidly increasing growth rates

$$r_1(s, n) = n - n^s \quad \text{and} \quad r_2(s, n) = n - 2^{(\log n)^s}.$$

The negative orders of scaled dimension use growth rates that increase much more slowly. In the -2^{nd} and -3^{rd} orders, we use

$$r_{-2}(s, n) = 2^{(\log n)^{1-s}} \quad \text{and} \quad r_{-3}(s, n) = 2^{2^{(\log \log n)^{1-s}}}.$$

Definition. The i^{th}*-order scaled* Δ*-dimension* of a class X, written $\dim^{(i)}_\Delta(X)$, is the infimum of all s where there exists a Δ-computable martingale d such that for all $A \in X$,

$$(\exists^\infty n)\ d(A \upharpoonright n) \geq 2^{r_i(s,n)}. \tag{8.1}$$

The i^{th}*-order scaled strong* Δ*-dimension* of X, written $\mathrm{Dim}^{(i)}_\Delta(X)$, is defined in the same way, instead requiring that (8.1) hold for all but finitely many n. We also define $\dim^{(i)}(X \mid \mathrm{E}) = \dim^{(i)}_{\mathrm{p}}(X \cap \mathrm{E})$, $\dim^{(i)}(X \mid \mathrm{EXP}) = \dim^{(i)}_{\mathrm{p}_2}(X \cap \mathrm{EXP})$, etc. (analogously to the definitions in Section 2).

The 0^{th}-order scaled dimension is the standard (unscaled) dimension. The other scaled dimensions have similar properties. For example, $0 \leq \dim^{(i)}_\Delta(X) \leq \mathrm{Dim}^{(i)}_\Delta(X) \leq 1$ and if $\dim^{(i)}_\Delta(X) < 1$, then X has Δ-measure 0. The following theorem states two important facts about the scaled dimensions.

Theorem 8.1. ([31]) *The scaled dimension* $\dim^{(i)}_\Delta(X)$ *is nondecreasing in the order* i*. There is at most one order* i *for which* $\dim^{(i)}_\Delta(X)$ *is not 0 or 1.*

In particular, the sequence of scaled dimensions must have one of the following four forms.

(i) For all i, $\dim^{(i)}_\Delta(X) = 0$.	(ii) For all i, $\dim^{(i)}_\Delta(X) = 1$.
(iii) There is an order i^* such that – $\dim^{(i)}_\Delta(X) = 0$ for all $i \leq i^*$ and – $\dim^{(i)}_\Delta(X) = 1$ for all $i > i^*$.	(iv) There is an order i^* such that – $\dim^{(i)}_\Delta(X) = 0$ for all $i < i^*$, – $0 < \dim^{(i^*)}_\Delta(X) < 1$, and – $\dim^{(i)}_\Delta(X) = 1$ for all $i > i^*$.

We find (iv) to be the most interesting case. Then i^* is the "best" order at which to measure the Δ-dimension of X because $\dim^{(i^*)}_\Delta(X)$ provides much more quantitative information about X than is provided by $\dim^{(i)}_\Delta(X)$ for $i \neq i^*$.

9 Circuit-Size Complexity II

Theorem 3.1 says that $\mathrm{SIZE}(\alpha\frac{2^n}{n})$ has dimension α in ESPACE. However, in complexity theory, circuit-size bounds such as $2^{\alpha n}$ and 2^{n^α} are of more interest than $\alpha\frac{2^n}{n}$. Theorem 3.1 implies that $\mathrm{SIZE}(2^{\alpha n})$ and $\mathrm{SIZE}(2^{n^\alpha})$ have dimension 0 for all $\alpha \in (0,1)$. This is where we find our first application of scaled dimension. The 1st and 2nd orders capture these circuit-size bounds.

Theorem 9.1. (Hitchcock, Lutz, and Mayordomo [31]) *For all $\alpha \in [0,1]$,*

$$\dim^{(1)}(\mathrm{SIZE}(2^{\alpha n}) \mid \mathrm{ESPACE}) = \dim^{(2)}(\mathrm{SIZE}(2^{n^\alpha}) \mid \mathrm{ESPACE}) = \alpha.$$

Proof sketch. We only sketch the upper bound for the 1st-order result. For each $n \geq 0$, let

$$C_n = \{v \in \{0,1\}^{2^n} \mid v \text{ has a circuit of size at most } 2^{\alpha n}\ \}.$$

Here we are viewing a $v \in \{0,1\}^{2^n}$ as the characteristic string of some subset of $\{0,1\}^n$. Let $\alpha' > \alpha$. A counting argument [31] shows that for some n_0, for all $n \geq n_0$, $|C_n| \leq 2^{2^{\alpha' n}}$.

We define a martingale inductively. We start with $d(w) = 1$ for all $w \in \{0,1\}^{\leq 2^{n_0}-1}$. Let $n \geq n_0$ and assume that $d(w)$ has been defined for all $w \in \{0,1\}^{\leq 2^n - 1}$. For any $w \in \{0,1\}^{2^n-1}$ and $u \in \{0,1\}^{\leq 2^n}$, we define

$$d(wu) = 2^{|u|} \cdot \frac{|\{v \in \{0,1\}^{2^n} \mid u \sqsubseteq v \text{ and } v \in C_n\}|}{|C_n|} d(w).$$

Then d is a pspace-computable martingale. The martingale can be viewed as betting its money according to how likely a random extension of the current characteristic string will have a circuit of size at most $2^{\alpha n}$. Observe that if $u \in C_n$, then

$$d(wu) = 2^{2^n} \cdot \frac{1}{|C_n|} d(w) \geq 2^{2^n - 2^{\alpha' n}} d(w)$$

by the bound on $|C_n|$ above. Letting $\alpha'' > \alpha'$, it follows that for all $A \in \mathrm{SIZE}(\alpha\frac{2^n}{n})$,

$$d(A \upharpoonright 2^n - 1) \geq 2^{(2^n-1)-(2^n-1)^{\alpha''}}$$

for all sufficiently large n. Therefore $\dim_{\mathrm{pspace}}(\mathrm{SIZE}(\alpha\frac{2^n}{n})) \leq \alpha''$. Since $\alpha'' > \alpha$ is arbitrarily close to α, $\dim^{(1)}(\mathrm{SIZE}(2^{\alpha n}) \mid \mathrm{ESPACE}) \leq \alpha$ follows. □

We remark that Theorem 9.1 also holds for strong scaled dimension. Also, parts of Theorem 3.3 extend to strong scaled dimension: Gu [21] showed that for every order $i \in \mathbb{Z}$, $\mathrm{Dim}^{(i)}(\mathrm{SIZE}(n^k) \mid \mathrm{EXP}) = 0$ for all k and $\mathrm{Dim}^{(i)}(\mathrm{P/poly} \mid \mathrm{E}_3) = 0$.

10 Polynomial-Time Degrees II

Recall from Section 4 that there is no small span theorem for resource-bounded dimension. The reason is that degrees have the same dimension as their lower spans. Does this also hold in scaled dimension? The proof of Theorem 4.1 extends to show that it does for the first few orders.

Theorem 10.1. (Hitchcock [28]) *For every* $A \in \mathrm{E}$ *and* $i \in \{-2, -1, 0, 1, 2\}$,

$$\dim^{(i)}(\mathrm{P_m}(A) \mid \mathrm{E}) = \dim^{(i)}(\deg^{\mathrm{p}}_{\mathrm{m}}(A) \mid \mathrm{E}).$$

There is an interesting contrast between the -2^{nd} and -3^{rd} orders. The proof of Theorem 4.1/10.1 uses a padding technique that breaks down in the -3^{rd} order. Here is a sketch of the reason why. Suppose we want to show that $\dim^{(i)}(\mathrm{P_m}(A)) \leq \dim^{(i)}(\deg^{\mathrm{p}}_{\mathrm{m}}(A))$. (The other inequality is trivial by monotonicity.) Letting $s > \dim^{(i)}(\deg^{\mathrm{p}}_{\mathrm{m}}(A))$, we have a martingale d that achieves

$$(\exists^{\infty} n) d(D \upharpoonright n) \geq 2^{r_i(s,n)} \tag{10.1}$$

for all $D \in \deg^{\mathrm{p}}_{\mathrm{m}}(A)$. We would like to obtain a martingale d' that achieves the same for all $B \in \mathrm{P_m}(A)$. To accomplish this the idea is to modify B slightly, encoding A very sparsely into it, to obtain a set $D \in \deg^{\mathrm{p}}_{\mathrm{m}}(A)$. Precisely, we let $k \geq 2$, use the padding function f defined by $f(x) = 0^{|x|^k - |x|}x$, and let

$$D = B - f(\{0,1\}^*) \cup f(A).$$

Now D is in the degree of A, so d succeeds on D as in (10.1). Since D is different from B only on the easily identifiable subset $R = f(\{0,1\}^*)$, it is easy to modify d to obtain a martingale d' that bets like d outside of R and does not bet at all on strings in R. Of course, d' may not do as well as d if d makes a substantial part of its winnings on strings in R. How much of a hit does d' take? Up to length n^k, there are $N_k = 2^{n^k+1} - 1$ strings while R has $N = 2^{n+1} - 1$ strings. Suppose that (10.1) holds for N_k, that is, $d(D \upharpoonright N_k) \geq 2^{r_i(s,N_k)}$. Then we have

$$d'(B \upharpoonright N_k) \geq 2^{r_i(s,N_k) - N},$$

since d bets like d' except for the strings in R. If N is much smaller than $r_i(s, N_k)$, this decrease will not be a problem for our dimension calculation. In the -2^{nd} order, we have

$$r_{-2}(s, N_k) = 2^{(\log N_k)^{(1-s)}} \approx 2^{n^{k(1-s)}}$$

and in the -3^{rd} order we have

$$r_{-3}(s, N_k) = 2^{2^{(\log\log N_k)^{(1-s)}}} \approx 2^{2^{(k \log n)^{(1-s)}}}.$$

For the -2^{nd} order, we can choose k large enough so that $N = o(r_{-2}(s, N_k))$. Therefore the N factor does not affect the growth of the martingale much, and d' shows that $\dim^{(-2)}(\mathrm{P_m}(A)) \leq \dim^{(-2)}(\deg^{\mathrm{p}}_{\mathrm{m}}(A))$. However, for the -3^{rd} order, $r_{-3}(s, N_k)$ is always $o(N)$ no matter how large k is. In this case, the N factor is very significant, and d' does not win fast enough to give a -3^{rd}-order dimension bound.

Not only does the above proof technique fail in the -3^{rd} order; it is impossible to extend Theorem 10.1 to $i = -3$. Indeed, the following small span theorem for scaled dimension yields that every degree has -3^{rd}-order dimension 0.

Theorem 10.2. (Hitchcock [28]) *For every $A \in \mathrm{E}$,*

$$\dim^{(1)}(\mathrm{P_m}(A) \mid \mathrm{E}) = 0 \quad \textit{or} \quad \dim^{(-3)}(\mathrm{P_m^{-1}}(A) \mid \mathrm{E}) = 0.$$

In particular, $\dim^{(-3)}(\deg^{\mathrm{p}}_{\mathrm{m}}(A) \mid \mathrm{E}) = 0$.

Theorem 10.2 was proved by extending the arguments of Juedes and Lutz [38] involving incompressibility by reductions. In another paper, Juedes and Lutz [39] used more probabilistic arguments to establish small span theorems within ESPACE for P/poly-Turing reductions. These are reductions computed by a nonuniform family of polynomial-size oracle circuits. This has also been extended to scaled dimension by Hitchcock, López-Valdés, and Mayordomo [29].

11 The Dimension of NP II

From Theorems 10.1 and 10.2, we learn that scaled dimension gives us two types of dimension for studying complexity classes such as NP. For example,

$$\dim^{(-2)}(\mathrm{NP} \mid \mathrm{E}) = \dim^{(-2)}(\mathrm{NPC} \mid \mathrm{E}),$$

while in the -3^{rd} order, NPC unconditionally has dimension 0.

The question of whether Turing completeness is different from many-one completeness for NP is an intriguing one that dates back to the first papers of Cook, Karp, and Levin on NP completeness. No hypothesis on NP was known to imply a separation of these completeness notions until Lutz and Mayordomo [54] used the "NP is not small" hypothesis from resource-bounded measure. A hypothesis on the p-dimension of NP such as "$\dim_{\mathrm{p}}(\mathrm{NP}) > 0$" or even "$\dim_{\mathrm{p}}(\mathrm{NP}) = 1$" is not known to imply this separation. However, recently it has been shown that a -3^{rd}-order positive-dimension hypothesis on NP suffices.

Theorem 11.1. (Hitchcock, Pavan, and Vinodchandran [34]) *If $\dim_{\mathrm{p}}^{(-3)}(\mathrm{NP}) > 0$, then Turing completeness and many-one completeness are different for* NP.

Impagliazzo and Moser [36] showed that the measure hypothesis on NP implies a full derandomization of Arthur-Merlin games: NP = AM. Their proof also relies on the almost-everywhere hardness guaranteed by $\mu_{\mathrm{p}}(\mathrm{NP}) \neq 0$. Weakening this to dimension seems challenging. However, taking the stronger -3^{rd}-order hypothesis achieves the derandomization when sublinear advice is given.

Theorem 11.2. (Hitchcock and Pavan [32]) *If $\dim_{\mathrm{p}}^{(-3)}(\mathrm{NP}) > 0$, then $\mathrm{AM} \subseteq \mathrm{NP}/n^{\epsilon}$ for all $\epsilon > 0$.*

12 Conclusion

Fractal dimensions in complexity classes are precise quantitative tools that bring complexity-theoretic and information-theoretic ideas together in new ways. We have surveyed some of the applications of these tools to date, but our survey is incomplete in two ways.

First, due to space constraints, our survey is far from comprehensive. We have omitted Moser's beautiful work on dimension in small complexity classes [62], and work of several authors on dimension and autoreducibility [2, 5, 20]. Even within the topics we have discussed, we have omitted many nice results.

Second, due to a conjectured time constraint [23], our survey omits research that has not yet occurred. We hope that your forthcoming theorems obsolesce our survey soon.

References

[1] E. Allender. When worlds collide: Derandomization, lower bounds, and Kolmogorov complexity. In *Proceedings of the 21st Conference on Foundations of Software Technology and Theoretical Computer Science*, pages 1–15. Springer-Verlag, 2001.

[2] K. Ambos-Spies, W. Merkle, J. Reimann, and F. Stephan. Hausdorff dimension in exponential time. In *Proceedings of the 16th IEEE Conference on Computational Complexity*, pages 210–217. IEEE Computer Society, 2001.

[3] K. Ambos-Spies, H.-C. Neis, and S. A. Terwijn. Genericity and measure for exponential time. *Theoretical Computer Science*, 168(1):3–19, 1996.

[4] K. B. Athreya, J. M. Hitchcock, J. H. Lutz, and E. Mayordomo. Effective strong dimension in algorithmic information and computational complexity. In *Proceedings of the 21st Annual Symposium on Theoretical Aspects of Computer Science*, pages 632–643. Springer-Verlag, 2004. To appear in *SIAM Journal on Computing*.

[5] R. Beigel, L. Fortnow, and F. Stephan. Infinitely-often autoreducible sets. In *Proceedings of the 14th Annual International Symposium on Algorithms and Computation*, pages 98–107. Springer-Verlag, 2003.

[6] C. H. Bennett and J. Gill. Relative to a random oracle A, $\mathrm{P}^A \neq \mathrm{NP}^A \neq \text{co-NP}^A$ with probability 1. *SIAM Journal on Computing*, 10:96–113, 1981.

[7] H. Buhrman and D. van Melkebeek. Hard sets are hard to find. *Journal of Computer and System Sciences*, 59(2):327–345, 1999.

[8] J. Cai and J. Hartmanis. On Hausdorff and topological dimensions of the Kolmogorov complexity of the real line. *Journal of Computer and Systems Sciences*, 49:605–619, 1994.

[9] J. J. Dai, J. I. Lathrop, J. H. Lutz, and E. Mayordomo. Finite-state dimension. *Theoretical Computer Science*, 310(1–3):1–33, 2004.

[10] J. L. Doob. Regularity properties of certain families of chance variables. *Transactions of the American Mathematical Society*, 47:455–486, 1940.

[11] R. Downey and D. Hirschfeldt. Algorithmic randomness and complexity. Book draft, 2005.

[12] G. A. Edgar. *Measure, Topology, and Fractal Geometry*. Springer-Verlag, 1990.

[13] G. A. Edgar. *Integral, Probability, and Fractal Measures*. Springer-Verlag, 1998.

[14] K. Falconer. *The Geometry of Fractal Sets*. Cambridge University Press, 1985.

[15] K. Falconer. *Techniques in Fractal Geometry*. John Wiley & Sons, 1997.

[16] K. Falconer. *Fractal Geometry: Mathematical Foundations and Applications*. John Wiley & Sons, second edition, 2003.

[17] H. Federer. *Geometric Measure Theory*. Springer-Verlag, 1969.

[18] L. Fortnow. Personal communication, 2001.

[19] L. Fortnow and J. H. Lutz. Prediction and dimension. *Journal of Computer and System Sciences*, 70(4):570–589, 2005.

[20] C. Glaßer, M. Ogihara, A. Pavan, A. Selman, and L. Zhang. Autoreducibility, mitoticity, and immunity. In *Proceedings of the 30th International Symposium on Mathematical Foundations of Computer Science*. Springer-Verlag, 2005.

[21] X. Gu. A note on dimensions of polynomial size circuits. Technical Report TR04-047, Electronic Colloquium on Computational Complexity, 2004.

[22] F. Hausdorff. Dimension und äußeres Maß. *Mathematische Annalen*, 79:157–179, 1919.

[23] S. W. Hawking. Chronology protection conjecture. *Physical Review D*, 46(2):603–611, 1992.

[24] J. M. Hitchcock. MAX3SAT is exponentially hard to approximate if NP has positive dimension. *Theoretical Computer Science*, 289(1):861–869, 2002.

[25] J. M. Hitchcock. *Effective Fractal Dimension: Foundations and Applications*. PhD thesis, Iowa State University, 2003.

[26] J. M. Hitchcock. Fractal dimension and logarithmic loss unpredictability. *Theoretical Computer Science*, 304(1–3):431–441, 2003.

[27] J. M. Hitchcock. Hausdorff dimension and oracle constructions. Technical Report TR04-072, Electronic Colloquium on Computational Complexity, 2004.

[28] J. M. Hitchcock. Small spans in scaled dimension. *SIAM Journal on Computing*, 34(1):170–194, 2004.

[29] J. M. Hitchcock, M. López-Valdés, and E. Mayordomo. Scaled dimension and the Kolmogorov complexity of Turing-hard sets. In *Proceedings of the 29th International Symposium on Mathematical Foundations of Computer Science*, pages 476–487. Springer-Verlag, 2004.

[30] J. M. Hitchcock and J. H. Lutz. Why computational complexity requires stricter martingales. In *Proceedings of the 29th International Colloquium on Automata, Languages, and Programming*, pages 549–560. Springer-Verlag, 2002. To appear in *Theory of Computing Systems*.

[31] J. M. Hitchcock, J. H. Lutz, and E. Mayordomo. Scaled dimension and nonuniform complexity. *Journal of Computer and System Sciences*, 69(2):97–122, 2004.

[32] J. M. Hitchcock and A. Pavan. Hardness hypotheses, derandomization, and circuit complexity. In *Proceedings of the 24th Conference on Foundations of Software Technology and Theoretical Computer Science*, pages 336–347. Springer-Verlag, 2004.

[33] J. M. Hitchcock and A. Pavan. Resource-bounded strong dimension versus resource-bounded category. *Information Processing Letters*, 95(3):377–381, 2005.

[34] J. M. Hitchcock, A. Pavan, and N. V. Vinodchandran. Partial bi-immunity, scaled dimension, and NP-completeness. Manuscript, 2005.

[35] J. M. Hitchcock and N. V. Vinodchandran. Dimension, entropy rates, and compression. In *Proceedings of the 19th IEEE Conference on Computational Complexity*, pages 174–183. IEEE Computer Society, 2004.

[36] R. Impagliazzo and P. Moser. A zero-one law for RP. In *Proceedings of the 18th IEEE Conference on Computational Complexity*, pages 43–47. IEEE Computer Society, 2003.

[37] R. Impagliazzo and A. Wigderson. Randomness vs. time: Derandomization under a uniform assumption. *Journal of Computer and System Sciences*, 63:672–688, 2001.

[38] D. W. Juedes and J. H. Lutz. The complexity and distribution of hard problems. *SIAM Journal on Computing*, 24(2):279–295, 1995.

[39] D. W. Juedes and J. H. Lutz. Completeness and weak completeness under polynomial-size circuits. *Information and Computation*, 125(1):13–31, 1996.

[40] J. Köbler and W. Lindner. On the resource bounded measure of P/poly. In *Proceedings of the 13th IEEE Conference on Computational Complexity*, pages 182–185. IEEE Computer Society, 1998.

[41] J. Köbler, U. Schöning, and J. Torán. On counting and approximation. *Acta Informatica*, 26:363–379, 1989.

[42] P. Lévy. Propriétés asymptotiques des sommes de variables indépendantes ou enchainées. *Journal des mathématiques pures et appliquées. Series 9*, 14(4):347–402, 1935.

[43] P. Lévy. *Théorie de l'Addition des Variables Aleatoires*. Gauthier-Villars, 1937 (second edition 1954).

[44] W. Lindner. On the polynomial time bounded measure of one-truth-table degrees and p-selectivity. Diplomarbeit, Technische Universität Berlin, 1993.

[45] M. López-Valdés and E. Mayordomo. Dimension is compression. In *Proceedings of the 30th International Symposium on Mathematical Foundations of Computer Science*. Springer-Verlag, 2005. To appear.

[46] J. H. Lutz. Category and measure in complexity classes. *SIAM Journal on Computing*, 19(6):1100–1131, 1990.

[47] J. H. Lutz. Almost everywhere high nonuniform complexity. *Journal of Computer and System Sciences*, 44(2):220–258, 1992.

[48] J. H. Lutz. The quantitative structure of exponential time. In L. A. Hemaspaandra and A. L. Selman, editors, *Complexity Theory Retrospective II*, pages 225–254. Springer-Verlag, 1997.

[49] J. H. Lutz. Resource-bounded measure. In *Proceedings of the 13th IEEE Conference on Computational Complexity*, pages 236–248. IEEE Computer Society, 1998.

[50] J. H. Lutz. Dimension in complexity classes. In *Proceedings of the Fifteenth Annual IEEE Conference on Computational Complexity*, pages 158–169. IEEE Computer Society, 2000.

[51] J. H. Lutz. Gales and the constructive dimension of individual sequences. In *Proceedings of the 27th International Colloquium on Automata, Languages, and Programming*, pages 902–913. Springer-Verlag, 2000. Revised as [53].

[52] J. H. Lutz. Dimension in complexity classes. *SIAM Journal on Computing*, 32(5):1236–1259, 2003.

[53] J. H. Lutz. The dimensions of individual strings and sequences. *Information and Computation*, 187(1):49–79, 2003.

[54] J. H. Lutz and E. Mayordomo. Cook versus Karp-Levin: Separating completeness notions if NP is not small. *Theoretical Computer Science*, 164(1–2):141–163, 1996.

[55] J. H. Lutz and E. Mayordomo. Twelve problems in resource-bounded measure. *Bulletin of the European Association for Theoretical Computer Science*, 68:64–80, 1999. Also in *Current Trends in Theoretical Computer Science: Entering the 21st Century*, pages 83–101, World Scientific Publishing, 2001.

[56] P. Martin-Löf. The definition of random sequences. *Information and Control*, 9:602–619, 1966.

[57] P. Matilla. *Geometry of Sets and Measures in Euclidean Spaces: Fractals and rectifiability.* Cambridge University Press, 1995.

[58] E. Mayordomo. *Contributions to the study of resource-bounded measure.* PhD thesis, Universitat Politècnica de Catalunya, 1994.

[59] E. Mayordomo. A Kolmogorov complexity characterization of constructive Hausdorff dimension. *Information Processing Letters*, 84(1):1–3, 2002.

[60] P. Moser. *Derandomization and Quantitative Complexity.* PhD thesis, Université de Genève, 2004.

[61] P. Moser. Generic density and small span theorem. In *Proceedings of the 15th International Symposium on Fundamentals of Computation Theory.* Springer-Verlag, 2005. To appear.

[62] P. Moser. Martingale families and dimension in P. Technical Report TR05-045, Electronic Colloquium on Computational Complexity, 2005.

[63] C. A. Rogers. *Hausdorff Measures.* Cambridge University Press, 1998. Originally published in 1970.

[64] B. Ya. Ryabko. Algorithmic approach to the prediction problem. *Problems of Information Transmission*, 29:186–193, 1993.

[65] B. Ya. Ryabko. The complexity and effectiveness of prediction problems. *Journal of Complexity*, 10:281–295, 1994.

[66] C. P. Schnorr. Klassifikation der Zufallsgesetze nach Komplexität und Ordnung. *Z. Wahrscheinlichkeitstheorie verw. Geb.*, 16:1–21, 1970.

[67] C. P. Schnorr. A unified approach to the definition of random sequences. *Mathematical Systems Theory*, 5:246–258, 1971.

[68] C. P. Schnorr. Zufälligkeit und Wahrscheinlichkeit. *Lecture Notes in Mathematics*, 218, 1971.

[69] C. P. Schnorr. Process complexity and effective random tests. *Journal of Computer and System Sciences*, 7:376–388, 1973.

[70] R. Shaltiel and C. Umans. Pseudorandomness for approximate counting and sampling. In *Proceedings of the 20th IEEE Conference on Computational Complexity*, pages 212–226. IEEE Computer Society, 2005.

[71] C. E. Shannon. The synthesis of two-terminal switching circuits. *Bell System Technical Journal*, 28:59–98, 1949.

[72] L. Staiger. A tight upper bound on Kolmogorov complexity and uniformly optimal prediction. *Theory of Computing Systems*, 31:215–29, 1998.

[73] L. J. Stockmeyer. On approximation algorithms for #P. *SIAM Journal on Computing*, 14:849–861, 1985.

[74] D. Sullivan. Entropy, Hausdorff measures old and new, and limit sets of geometrically finite Kleinian groups. *Acta Mathematica*, 153:259–277, 1984.

[75] C. Tricot. Two definitions of fractional dimension. *Mathematical Proceedings of the Cambridge Philosophical Society*, 91:57–74, 1982.

[76] D. van Melkebeek. The zero-one law holds for BPP. *Theoretical Computer Science*, 244(1–2):283–288, 2000.

[77] J. Ville. *Étude Critique de la Notion de Collectif.* Gauthier–Villars, Paris, 1939.