Resource-Bounded Strong Dimension versus Resource-Bounded Category

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Abstract

Classically it is known that any set with packing dimension less than 1 is meager in the sense of Baire category. We establish a resource-bounded extension: if a class X has Δ-strong dimension less than 1, then X is Δ-meager. This has the applications of explaining some of Lutz's simultaneous Δ-meager, Δ-measure 0 results and providing a new proof of a Gu's strong dimension result on infinitely-often classes.

1 Introduction

The most common mathematical notions of size and dimension now have resource-bounded versions that are useful for complexity classes. We use Δ to denote a resource bound such as p (polynomial time) or pspace (polynomial space).

- *Resource-Bounded Category* [7]: Extension of *Baire category.* Complexity classes may be Δ-meager or Δ-comeager (or neither).
- *Resource-Bounded Measure* [8]: Extension of *Lebesgue measure.* The Δ-measure of a complexity class X is denoted $\mu_\Delta(X)$. A class X may have $\mu_\Delta(X) = 0$ or $\mu_\Delta(X) = 1$ (or neither, in which case the class is called not Δ-measurable).
- *Resource-Bounded Dimension* [9]: Extension of *Hausdorff dimension* [6]. Each complexity class X has a Δ-dimension $\dim_\Delta(X)$ that is always a real number in [0,1].
- *Resource-Bounded Strong Dimension* [3]: Extension of *packing dimension* [12, 11]. Each complexity class X has a Δ-strong dimension $\mathrm{Dim}_\Delta(X)$ that is always a real number in [0,1].

In general, resource-bounded category and resource-bounded measure are incomparable: Δ-meager does not imply Δ-measure 0, and vice versa. Regarding measure versus the two notions of dimension, the following hold for every class X:

$$\dim_\Delta(X) \leq \mathrm{Dim}_\Delta(X)$$

and

$$\dim_\Delta(X) < 1 \Rightarrow \mu_\Delta(X) = 0.$$

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In particular, it follows that if the Δ-strong dimension of X is less than 1, then X has Δ-measure 0. We show that $\mathrm{Dim}_\Delta(X) < 1$ also implies X is Δ-meager. This is an extension of the analogous relationship between packing dimension and Baire category (see Edgar [4]).

We give two applications of this result:

- An explanation of why some complexity classes in the work of Lutz [7] have Δ-measure 0 and are also Δ-meager. It is because they have Δ-strong dimension less than 1 (Gu [5]).
- A new category-based proof of Gu's result regarding the strong dimension of infinitely-often classes [5].

Section 2 contains preliminaries and background on category, measure, and dimension. Our main theorem is presented in section 3. The applications are given in section 4.

2 Category, Measure, and Dimension

The *Cantor space* $\mathbf{C}$ is the set of all infinite binary sequences. A *language* (or *decision problem*) is a subset of $\{0,1\}^*$. We identify each language with the element of Cantor space that is its characteristic sequence according to the standard enumeration of $\{0,1\}^*$. In this way, complexity classes (sets of languages) are viewed as subsets of Cantor space.

A *constructor* is a function $\delta : \{0,1\}^* \to \{0,1\}^*$. The *result* of a constructor is the unique sequence $R(\delta) \in \mathbf{C}$ that extends $\delta^{(n)}(\lambda)$ for all n. (Here λ is the empty string.)

Throughout this paper, Δ denotes a *resource bound* [8]. Examples of Δ include:

$$\begin{aligned}
\text{all} &= \{f \mid f : \{0,1\}^* \to \{0,1\}^*\} \\
\text{p} &= \{f \mid f \text{ is polynomial-time computable}\} \\
\text{p}_2 &= \{f \mid f \text{ is quasipolynomial-time computable}\} \\
\text{pspace} &= \{f \mid f \text{ is polynomial-space computable}\} \\
\text{comp} &= \{f \mid f \text{ is computable}\}
\end{aligned}$$

For a resource bound Δ, we define the class

$$R(\Delta) = \{R(\delta) \mid \delta \in \Delta \text{ is a constructor}\}.$$

Then $R(\text{all}) = \mathbf{C}$, $R(\text{p}) = \text{E}$, $R(\text{p}_2) = \text{EXP}$, $R(\text{pspace}) = \text{ESPACE}$, and $R(\text{comp}) = \text{DEC}$. Each resource bound Δ yields notions of resourced-bounded category, measure, and dimension that work within the class $R(\Delta)$. We now review these concepts.

2.1 Category

Baire category classifies sets into two types: *first category* and *second category*. First category sets are also commonly called *meager*. A set is meager if it is a countable union of nowhere dense sets. An equivalent definition comes from Banach-Mazur games.

Let $X \subseteq \mathbf{C}$ and let Γ_{I} and Γ_{II} be two classes of functions. In the *Banach-Mazur game* $G[X; \Gamma_{\mathrm{I}}, \Gamma_{\mathrm{II}}]$ there are two players I and II. A *strategy* in the game is a constructor. In a play of the game, player I chooses a strategy $g \in \Gamma_{\mathrm{I}}$ and player II chooses a strategy $h \in \Gamma_{\mathrm{II}}$. The *result* of this play is the sequence $R(g,h) = R(h \circ g)$. Intuitively, the result is the sequence obtained when the two players start with the empty string and take turns extending it with their strategies. A *winning strategy* for player II is a strategy $h \in \Gamma_{\mathrm{II}}$ such that for every $g \in \Gamma_{\mathrm{I}}$, $R(g,h) \notin X$.

Theorem 2.1. (Banach and Mazur) *A class $X \subseteq \mathbf{C}$ is meager if and only if player II has a winning strategy in the game $G[X; \text{all}, \text{all}]$.*

Resource-bounded category [7] is defined by requiring player II's winning strategy to be computable within a resource bound.

Definition. Let $X \subseteq \mathbf{C}$.

1. X is Δ-*meager* if player II has a winning strategy in the game $G[X; \text{all}, \Delta]$.
2. X is Δ-*comeager* if X^c is Δ-meager.
3. X is *meager in* $R(\Delta)$ if $X \cap R(\Delta)$ is Δ-meager.
4. X is *comeager in* $R(\Delta)$ if X^c is meager in $R(\Delta)$.

The *resource-bounded Baire category theorem* [7] tells us that $R(\Delta)$ is not Δ-meager.

2.2 Measure

A *martingale* is a function $d : \{0,1\}^* \to [0, \infty)$ satisfying the averaging condition

$$d(w) = \frac{d(w0) + d(w1)}{2}$$

for all $w \in \{0,1\}^*$. We say d *succeeds on* a sequence $S \in \mathbf{C}$ if

$$\limsup_{n \to \infty} d(S \restriction n) = \infty.$$

(Here $S \restriction n$ is the length n prefix of S.) The *success set* of d is

$$S^\infty[d] = \{S \in \mathbf{C} \mid d \text{ succeeds on } S\}.$$

Ville used martingales to give an equivalent definition of Lebesgue measure 0.

Theorem 2.2. (Ville [13]) *A class $X \subseteq \mathbf{C}$ has Lebesgue measure 0 if and only if there is a martingale d with $X \subseteq S^\infty[d]$.*

Resource-bounded measure [8] arises from putting resource bounds on the martingales. We say that $d : \{0,1\}^* \to [0, \infty)$ is Δ-*computable* if there is an approximation $\hat{d} : \mathbb{N} \times \{0,1\}^* \to \mathbb{Q}$ such that $|\hat{d}(r, w) - d(w)| \leq 2^{-r}$ for all $r \in \mathbb{N}, w \in \{0,1\}^*$ and $\hat{d} \in \Delta$ (with r encoded in unary and the outputs encoded in binary).

Definition. Let $X \subseteq \mathbf{C}$.

1. X has Δ-*measure 0*, written $\mu_\Delta(X) = 0$, if there is a Δ-computable martingale d with $X \subseteq S^\infty[d]$.
2. X has Δ-*measure 1*, written $\mu_\Delta(X) = 1$, if $\mu_\Delta(X^c) = 0$.
3. X has *measure 0 in* $R(\Delta)$, written $\mu(X \mid R(\Delta)) = 0$, if $\mu_\Delta(X \cap R(\Delta)) = 0$.
4. X has *measure 1 in* $R(\Delta)$, written $\mu(X \mid R(\Delta)) = 1$, if $\mu_\Delta(X^c \mid R(\Delta)) = 0$.

The *resource-bounded measure conservation theorem* [8] tells us that $R(\Delta)$ does not have Δ-measure 0.

2.3 Dimension and Strong Dimension

The most commonly used fractal dimension is the *Hausdorff dimension* $\dim_{\mathrm{H}}(X)$. Lutz used success sets of functions called gales to characterize Hausdorff dimension. Let $s \geq 0$ be a real number. An *s-gale* is a function $d : \{0,1\}^* \to [0,\infty)$ satisfying the condition

$$d(w) = \frac{d(w0) + d(w1)}{2^s}$$

for all $w \in \{0,1\}^*$. Note that a martingale is a 1-gale. "Succeeds on" and "success set" are defined for s-gales in the same way as for martingales.

Theorem 2.3. (Lutz [9]) *For every* $X \subseteq \mathbf{C}$,

$$\dim_{\mathrm{H}}(X) = \inf\{s \,|\, \textit{there is an } s\textit{-gale } d \textit{ with } X \subseteq S^\infty[d]\}.$$

Another common fractal dimension is the *packing dimension* $\dim_{\mathrm{P}}(X)$. This has an analogous gale characterization using the notion of strong success. An s-gale d *succeeds strongly on* a sequence $S \in \mathbf{C}$ if

$$\liminf_{n\to\infty} d(S \upharpoonright n) = \infty.$$

The *strong success set* of d is

$$S^\infty_{\mathrm{str}}[d] = \{S \in C \mid d \text{ succeeds strongly on } S\}.$$

Theorem 2.4. (Athreya, Hitchcock, Lutz, and Mayordomo [3]) *For every* $X \subseteq \mathbf{C}$,

$$\dim_{\mathrm{P}}(X) = \inf\{s \,|\, \textit{there is an } s\textit{-gale } d \textit{ with } X \subseteq S^\infty_{\mathrm{str}}[d]\}.$$

Based on Theorems 2.3 and 2.4, resource-bounded dimension and resource-bounded strong dimension are defined as extensions of Hausdorff dimension and packing dimension, respectively, by requiring the gales to be computable within a resource bound.

Definition. Let $X \subseteq \mathbf{C}$.

1. The Δ-*dimension of* X is

$$\dim_\Delta(X) = \inf\{s \,|\, \text{there is a } \Delta\text{-computable } s\text{-gale } d \text{ with } X \subseteq S^\infty[d]\}.$$

2. The Δ-*strong dimension of* X is

$$\mathrm{Dim}_\Delta(X) = \inf\{s \,|\, \text{there is a } \Delta\text{-computable } s\text{-gale } d \text{ with } X \subseteq S^\infty_{\mathrm{str}}[d]\}.$$

3. The *dimension of* X *in* $R(\Delta)$ is $\dim(X \mid R(\Delta)) = \dim_\Delta(X \cap R(\Delta))$.

4. The *strong dimension of* X *in* $R(\Delta)$ is $\mathrm{Dim}(X \mid R(\Delta)) = \mathrm{Dim}_\Delta(X \cap R(\Delta))$.

We say that an s-gale d is *exactly* Δ-*computable* if the range of d is rational and the values can be computed by a function in Δ. The *exact computation lemma* [9] tells us that we may restrict to exactly computable s-gales in the above definitions.

Proposition 2.5. ([9, 3]) *Let* $X \subseteq \mathbf{C}$.

1. $0 \leq \dim_\Delta(X) \leq \mathrm{Dim}_\Delta(X) \leq 1$.

2. *If* $\dim_\Delta(X) < 1$, *then* X *has* Δ-*measure 0.*

3 Main Theorem

It is known classically that if the packing dimension $\dim_{\mathrm{P}}(X) < 1$, then X is meager (see Edgar [4, page 65]). We now establish the resource-bounded extension of this fact.

Theorem 3.1. *Let $X \subseteq \mathbf{C}$.*

1. *If $\mathrm{Dim}_\Delta(X) < 1$, then X is Δ-meager.*
2. *If $\mathrm{Dim}(X \mid R(\Delta)) < 1$, then X is meager in $R(\Delta)$.*

Proof. Part 2 is immediate from part 1. Assume the hypothesis of part 1. Then for some $s < 1$, there is an exactly-Δ-computable s-gale d with $X \subseteq S^\infty_{\mathrm{str}}[d]$.

Let $t = \left\lceil \frac{s}{1-s} \right\rceil$. For each $w \in \{0,1\}^*$, we inductively construct an extension w' of w by the following algorithm.

$w' := w$.
for $i = 1$ to $t|w|$
  if $d(w'0) \le d(w'1)$
    $w' := w'0$.
  else
    $w' := w'1$.

Because d is an s-gale, the average of $d(w'0)$ and $d(w'1)$ is $2^{s-1}d(w')$. One of $d(w'0)$ and $d(w'1)$ must be no more than this average, so $d(w')$ decreases by multiplicative factor of 2^{s-1} (or a smaller factor) each iteration of the for-loop. Therefore, $d(w') \le 2^{(s-1)t|w|}d(w)$ at the end.

We define a constructor $h : \{0,1\}^* \to \{0,1\}^*$ by

$$h(w) = w'.$$

Then $h \in \Delta$ by the above algorithm. For each $w \in \{0,1\}^*$,

$$d(h(w)) \le 2^{(s-1)t|w|}d(w) \le 2^{-s|w|}d(w).$$

Also, $d(w) \le 2^{s|w|}d(\lambda)$ because d is an s-gale, so we have

$$d(h(w)) \le d(\lambda) \tag{3.1}$$

for every $w \in \{0,1\}^*$.

Since $h \in \Delta$, it suffices to show that h always wins the Banach-Mazur game $G[X; \mathrm{all}, \Delta]$ for player II. Let g be any constructor. Let $R(g,h)$ be the sequence built when g and h are played against each other. We need to show that $R(g,h) \notin X$. For this we can show $R(g,h) \notin S^\infty_{\mathrm{str}}[d]$ since $X \subseteq S^\infty_{\mathrm{str}}[d]$.

Define $w_0 = \lambda$ and $w_n = h(g(w_{n-1}))$ for all $n \ge 1$. Then each w_n is a prefix of $R(g,h)$ and for every $n \ge 1$,

$$d(w_n) = d(h(g(w_{n-1}))) \le d(\lambda)$$

by (3.1). Therefore

$$\liminf_{n\to\infty} d(R(g,h){\restriction}n) \le d(\lambda),$$

i.e., d does not succeed strongly on $R(g,h)$. □

We remark that Theorem 3.1 does not extend to *resource-bounded genericity*, a different notion of resource-bounded category. For example, we might ask if strong $\mathrm{Dim}_{\mathrm{p}}(X) < 1$ implies that X has no p-generics. This is false because there are sparse n^2-generics [2] but the class of sparse languages has strong p-dimension 0.

4 Corollaries

In general, Δ-measure 0 and Δ-meager are incomparable properties. For example, Mayordomo [10] showed that the class of non-P-bi-immune languages has p-measure 0 but is not p-meager. There are also examples of classes that are Δ-meager which do not have Δ-measure 0 [1]. However, Lutz [7, 8] showed that several classes both have Δ-measure 0 and are Δ-meager. Proposition 2.5 and Theorem 3.1 give us the following corollary which along with recent work of Gu [5] provides further explanation of Lutz's results.

Corollary 4.1. *Let* $X \subseteq \mathbf{C}$.

1. *If* $\mathrm{Dim}_\Delta(X) < 1$, *then* X *has* Δ*-measure 0 and is* Δ*-meager.*
2. *If* $\mathrm{Dim}(X \mid R(\Delta)) < 1$, *then* X *has measure 0 in* $R(\Delta)$ *and is meager in* $R(\Delta)$.

For example, Lutz showed that for each constant c, the circuit-size complexity class $\mathrm{SIZE}(n^c)$ has p_2-measure 0 and is p_2-meager. Gu showed $\mathrm{Dim}_{\mathrm{p}_2}(\mathrm{SIZE}(n^c)) = 0$. By Corollary 4.1, this yields a new proof of Lutz's simultaneous measure 0 and meager result. Similarly, Lutz showed that P/poly has p_3-measure 0 and is p_3-meager; Gu showed that $\mathrm{Dim}_{\mathrm{p}_3}(\mathrm{P/poly}) = 0$.

However, we remark that the converse of Corollary 4.1 does not hold in general. For example, the class $\mathrm{SIZE}(\frac{2^n}{n})$ has pspace-measure 0 and is pspace-meager [7], but it has pspace-dimension 1 [9].

For a class X of languages, define

$$\text{io-}X = \{A \subseteq \{0,1\}^* \mid (\exists B \in X)(\exists^\infty n) A_{=n} = B_{=n}\}.$$

(Here $A_{=n} = A \cap \{0,1\}^n$.) Gu [5] showed that if X contains the empty language, then $\dim(\text{io-}X \mid R(\Delta)) \geq 1/2$ and $\mathrm{Dim}(\text{io-}X \mid R(\Delta)) = 1$ for every Δ. Theorem 3.1 along with the resource-bounded Baire category theorem provides a simpler proof of (a minor extension of) the latter fact.

Corollary 4.2. (Gu [5]) *If* $X \cap R(\Delta) \neq \emptyset$, *then* $\mathrm{Dim}(\text{io-}X \mid R(\Delta)) = 1$.

Proof. Let $B \in X \cap R(\Delta)$. Then

$$E = \{A \subseteq \{0,1\}^* \mid (\exists^\infty n) A_{=n} = B_{=n}\}$$

is a subclass of io-X. Because $B \in R(\Delta)$, it can be shown that E is Δ-comeager. In particular, E^c is meager in $R(\Delta)$.

Suppose that $\mathrm{Dim}(\text{io-}X \mid R(\Delta)) < 1$. Then io-$X$ is meager in $R(\Delta)$ by Theorem 3.1, so E is also meager in $R(\Delta)$. But since the meager sets are closed under union, the resource-bounded Baire category theorem tells us we cannot have both E and E^c meager in $R(\Delta)$. □

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