Translating Sequent Proofs into English

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0.1 Translating Sequent Proofs into English

Gentzen[1] devised the sequent proof system to reflect how proofs are done in ordinary mathematics. The formal sequent proof is a tree structure and we could easily write and algorithm that would recursively translate sequent proofs into English. The rules for such a transformation are given in the following sections.

0.1.1 Axiom Rules

The axiom rule has the form:

$$\frac{}{\Gamma_1, \phi, \Gamma_2 \vdash \Delta_1, \phi, \Delta_2}\ (\text{Ax})$$

To translate an instance ot the axiom rule, we say:

> "But we know ϕ is true since we have assumed it."

or

> "ϕ holds since we assumed ϕ to be true and so we are done."

The $\perp$ on the left axiom is formally given as follows:

$$\frac{}{\Gamma_1, \perp, \Gamma_2 \vdash \Delta}\ (\perp\text{Ax})$$

We say:

> "But now we have assumed false and the theorem is true."

or

> "But now, we have derived a contradiction and the theorem is true."

[1] Gentzen, Gerhard

0.1.2 Conjunction Rules

The right conjunction rule is:

$$\frac{\Gamma \vdash \Delta_1, \phi, \Delta_2 \qquad \Gamma \vdash \Delta_1, \psi, \Delta_2}{\Gamma \vdash \Delta_1, (\phi \wedge \psi), \Delta_2}\ (\wedge\text{R})$$

We say:

> "To show $\phi \wedge \psi$ there are two cases, (case 1.) $\langle\langle$*insert translated proof of the left branch here*$\rangle\rangle$ (case 2.) $\langle\langle$*insert translated proof of the right branch here.*$\rangle\rangle$."

Or we say:

> "To show $\phi \wedge \psi$ we must show ϕ and we must show ψ. To see that ϕ holds: $\langle\langle$*insert translated proof of left branch here*$\rangle\rangle$ This completes the proof of ϕ. To see that ψ holds: $\langle\langle$*insert translated proof of right branch here.*$\rangle\rangle$ This completes the proof of $\phi \wedge \psi$."

The left conjunction rule is stated as follows:

$$\frac{\Gamma_1, \phi, \psi, \Gamma_2 \vdash \Delta}{\Gamma_1, (\phi \wedge \psi), \Gamma_2 \vdash \Delta}\ (\wedge\text{L})$$

To translate this rule, we say:

> "Since we have assumed $\phi \wedge \psi$, we assume ϕ and we assume ψ. $\langle\langle$*Insert translated proof of the premise here.*$\rangle\rangle$"

0.1.3 Disjunction

The formal rule for a disjunction on the right is:

$$\frac{\Gamma \vdash \Delta_1, \phi, \psi, \Delta_2}{\Gamma, \vdash \Delta_1, (\phi \vee \psi), \Delta_2}\ (\vee\text{R})$$

To translate, we say:

> "To show $\phi \vee \psi$ we must either show ϕ or show ψ. $\langle\langle$*Insert translated proof of the premise here.*$\rangle\rangle$"

The sequent proof rule for disjunction on the left is:

$$\frac{\Gamma_1, \phi, \Gamma_2 \vdash \Delta \qquad \Gamma_1, \psi, \Gamma_2 \vdash \Delta}{\Gamma_1, (\phi \vee \psi), \Gamma_2 \vdash \Delta}\ (\vee\text{L})$$

To translate, we say:

> "Since we know $\phi \vee \psi$ we proceed by cases: suppose ϕ is true, then $\langle\langle$*insert translated proof from the left branch here*$\rangle\rangle$. On the other hand, if ψ holds: $\langle\langle$*insert translated proof from right branch here.*$\rangle\rangle$"

or, we say:

> "Since $\phi \vee \psi$ holds, we proceed consider the two cases: (case 1. ϕ holds:) $\langle\langle$*insert translated proof from the left branch here*$\rangle\rangle$. (case 2. ψ holds:) $\langle\langle$*insert translated proof from right branch here.*$\rangle\rangle$

0.1.4 Implication Rules

The formal rule for an implication on the right is:

$$\frac{\Gamma, \phi \vdash \Delta_1, \psi, \Delta_2}{\Gamma \vdash \Delta_1, (\phi \Rightarrow \psi), \Delta_2} \ (\Rightarrow\text{R})$$

We say:

> "To prove $\phi \Rightarrow \psi$, assume ϕ and show ψ, $\langle\langle$*insert translated proof of the subgoal here.*$\rangle\rangle$."

or simply say:

> "Assume ϕ and show one of Δ, $\langle\langle$*insert translated proof of the subgoal here.*$\rangle\rangle$."

The formal rule for an implication on the left is:

$$\frac{\Gamma_1, \Gamma_2 \vdash \phi, \Delta \qquad \Gamma_1, \psi, \Gamma_2 \vdash \Delta}{\Gamma_1, (\phi \Rightarrow \psi), \Gamma_2 \vdash \Delta} \ (\Rightarrow\text{L})$$

We say:

> "Since we have assumed $\phi \Rightarrow \psi$, if we show ϕ we can assume ψ to show one of[2] Δ. To see that ϕ holds: $\langle\langle$*insert translated proof of left branch here.*$\rangle\rangle$ Now, we assume ψ. $\langle\langle$*Insert translated proof of right branch here.*$\rangle\rangle$"

0.1.5 Negation

The formal rule for a negation on the right is:

$$\frac{\Gamma, \phi \vdash \Delta_1, \Delta_2}{\Gamma \vdash \Delta_1, \neg\phi, \Delta_2} \ (\neg\text{R})$$

To translate, we say:

[2] If Δ is a single formula drop the phrase "one of". If Δ is empty say "False" instead.

"Assume ϕ. $\langle\langle$*Insert translated proof of premise here.*$\rangle\rangle$"

or we say

"Since we must show $\neg\phi$, assume ϕ. $\langle\langle$*Insert translated proof of premise here.*$\rangle\rangle$"

The formal rule for a negation on the left is:

$$\frac{\Gamma_1, \neg\phi, \Gamma_2 \vdash \Delta}{\Gamma_1, \Gamma_2 \vdash \phi, \Delta} \ (\neg\text{L})$$

To translate, we say:

"Since we have assumed $\neg\phi$, we show ϕ. $\langle\langle$*Insert translated proof of premise here.*$\rangle\rangle$"

or, we say:

"Since we know $\neg\phi$, we prove ϕ. $\langle\langle$*Insert translated proof of the premise here.*$\rangle\rangle$"

0.1.6 Universal Quantifier

The sequent proof rule for a $\forall$ on the right is:

$$\frac{\Gamma \vdash \Delta_1, \phi[x := y], \Delta_2}{\Gamma \vdash \Delta_1, \forall x.\phi, \Delta_2} \ (\forall\text{R}) \qquad \text{where variable } y \text{ is not free in any formula of } (\Gamma \cup \Delta_1 \cup \{\forall x.\phi\} \cup \Delta_2).$$

We say:

"To prove $\forall x.\phi$, choose an arbitrary y and show $\phi[x := y]$ [3]. $\langle\langle$*Insert translated proof of the premise here.*$\rangle\rangle$"

or, we simply say:

"Pick an arbitrary y and show $\phi[x := y]$. $\langle\langle$*Insert translated proof of the premise here.*$\rangle\rangle$"

The formal rule for $\forall$ on the left says:

$$\frac{\Gamma_1, \phi[x := t], \Gamma_2 \vdash \Delta}{\Gamma_1, \forall x.\phi, \Gamma_2 \vdash \Delta} \ (\forall\text{L}) \qquad \text{where } t \in \mathcal{T}.$$

To translate this rule, we say:

" Since we know that for every x, ϕ is true, assume $\phi[x := t]$. $\langle\langle$*Insert translated proof of premise here.*$\rangle\rangle$"

or, we say:

"Assume $\phi[x := t]$."

[3] In this rule, and those that follow, we say $\phi[x := y]$ to be the formula that results from the substitution of y fopr x in phi, *i.e.* actually do the substitution before writing the formula in your proof.

0.1.7 Existential Quantifiers

The rule for $\exists$ on the right is:

$$\frac{\Gamma \vdash \Delta_1, \phi[x := t], \Delta_2}{\Gamma \vdash \Delta_1, \exists x.\phi, \Delta_2} \ (\exists\text{R}) \qquad \text{where } t \in \mathcal{T}.$$

We say:

> "Let t be the witness for x in $\exists x.\phi$. We must show $\phi[x := t]$.$\langle\langle$*Insert translated proof of the premise here.*$\rangle\rangle$"

or, we say

> "To show $\exists x.\phi$, we choose the witness t and show $\phi[x := t]$. $\langle\langle$*Insert translated proof of the premise here.*$\rangle\rangle$"

The rule for $\exists$ on the left is:

$$\frac{\Gamma_1, \phi[x := y], \Gamma_2 \vdash \Delta}{\Gamma_1, \exists x.\phi, \Gamma_2 \vdash \Delta} \ (\exists\text{L}) \qquad \text{where variable } y \text{ is not free in any formula of } (\Gamma \cup \Delta_1 \cup \{\exists x.\phi\} \cup \Delta_2).$$

We say:

> "Since we know $\exists x.\phi$, pick an arbitrary element of the domain of discourse, call it y, and assume $\phi[x := y]$. $\langle\langle$*Insert translated proof of the premise here.*$\rangle\rangle$"

or, we say:

> "we know ϕ, holds for arbitrate x, so assume $\phi[x := y]$. $\langle\langle$*Insert translated proof of the premise here.*$\rangle\rangle$"

Examples We follow this discussion with a couple of examples.

Example 0.1. Consider the following sequent proof from **Example ??**.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{P(z) \vdash P(z)} \ (\text{Ax})}{P(z) \vdash \exists y.P(y)} \ (\exists\text{R})}{\forall x.P(x) \vdash \exists y.P(y)} \ (\forall\text{L})}{\vdash \forall x.P(x) \Rightarrow \exists y.P(y)} \ (\Rightarrow\text{R})$$

We apply the translation algorithm step by step working from the bottom up. Since the first rule is the $\Rightarrow R$ rule, we start as follows:

> To prove $\forall x.P(x) \Rightarrow \exists y.P(y)$ assume $\forall x.P(x)$ and show $\exists y.P(y)$. $\langle\langle$*Insert translated proof here.*$\rangle\rangle$

Since the next rule in the proof (working toward the top) is $\forall R$ we extend the translation as follows.

> To prove $\forall x.P(x) \Rightarrow \exists y.P(y)$ assume $\forall x.P(x)$ and show $\exists y.P(y)$. Since we know $\forall x.P(x)$ is true, we assume $P(z)$. $\langle\langle$*More translation goes here*$\rangle\rangle$

Now we apply the $\exists R$ translation rule, filling in more of the proof.

> To prove $\forall x.P(x) \Rightarrow \exists y.P(y)$ assume $\forall x.P(x)$ and show $\exists y.P(y)$. Since we know $\forall x.P(x)$ is true, we assume $P(z)$. Let z be the witness for y in $\exists y.P(y)$. We must show $P(z)$. $\langle\langle$*More translation goes here.*$\rangle\rangle$

Finally, we finish the translation by inserting the translation of the axiom rule.

> To prove $\forall x.P(x) \Rightarrow \exists y.P(y)$ assume $\forall x.P(x)$ and show $\exists y.P(y)$. Since we know $\forall x.P(x)$ is true, we assume $P(z)$. Let z be the witness for y in $\exists y.P(y)$. We must show $P(z)$. $P(z)$ holds since we assumed $P(z)$ to be true and so we are done.

Example 0.2. Consider the proof from **Example ??**.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P(x) \vdash P(x), S}\,\text{Ax}}{\vdash P(x), P(x) \Rightarrow S}\,{\Rightarrow}\text{R}}{\vdash P(x), \exists x.(P(x) \Rightarrow S)}\,(\exists\text{R})}{\vdash \forall x.P(x), \exists x.(P(x) \Rightarrow S)}\,(\forall\text{R}) \qquad \dfrac{\dfrac{\dfrac{}{S, P(z) \vdash \Rightarrow S}\,(\text{AX})}{S \vdash P(z) \Rightarrow S}\,({\Rightarrow}\text{R})}{S \vdash \exists x.(P(x) \Rightarrow S)}\,(\exists\text{R})}{(\forall x.P(x)) \Rightarrow S \vdash \exists x.(P(x) \Rightarrow S)}\,({\Rightarrow}\text{L})$$

Here is the resulting English translation.

> Since we have assumed $(\forall x.P(x)) \Rightarrow S$, we show $(\forall x.P(x))$ and separately assume S to show $\exists x.(P(x) \Rightarrow S)$. To see that $\forall x.P(x)$ holds: Pick an arbitrary x and show $P(x)$. Let x be the witness for x in $\exists x.(P(x) \Rightarrow S)$. We must show $P(x) \Rightarrow S$. Assume $P(x)$ and show one of $S, P(x)$. But we know $P(x)$ is true since we have assumed it.
>
> Now, we assume S. To show $\exists x.(P(x) \Rightarrow S)$, we choose the witness z and show $P(z) \Rightarrow S$. Assume $P(z)$ and show S. Be we know S is true because we have assumed it.