HW 12 **Prof. Caldwell**
Due: 8 October 2009 **COSC 2300**

Here are definitions of equality for sets, subset and the union and intersection and union operations:

$$\begin{array}{rcl} x \notin A & \stackrel{\text{def}}{=} & \neg(x \in A) \\ A \subseteq B & \stackrel{\text{def}}{=} & \forall x. x \in A \Rightarrow x \in B \\ A = B & \stackrel{\text{def}}{=} & \forall x. x \in A \Leftrightarrow x \in B \end{array}$$

The following two rules allow definitions to be unfolded in a sequent.

$$\frac{\Gamma_1, \psi, \Gamma_2 \vdash \Delta}{\Gamma_1, \phi, \Gamma_2 \vdash \Delta} \text{ (def of } op) \qquad \text{where } \phi \stackrel{\text{def}}{=} \psi \text{ is a definition of } op.$$

$$\frac{\Gamma \vdash \Delta_1, \psi, \Delta_2}{\Gamma \vdash \Delta_1, \phi, \Delta 2} \text{ (def of } op) \qquad \text{where } \phi \stackrel{\text{def}}{=} \psi \text{ is a definition of } op.$$

So, an instance of the definition rule to unfold $\subseteq$ would appear as follows:

$$\frac{\forall x. x \notin \emptyset \vdash \forall x.(x \in \emptyset \Rightarrow x \in A)}{\forall x. x \notin \emptyset \vdash \emptyset \subseteq A} \text{ (def of } \subseteq)$$

Also, for any nonlogical axiom or previously proved theorem ϕ referred to by the name *name*, the following rules holds.

$$\frac{\phi, \Gamma \vdash \Delta}{\Gamma \vdash \Delta} \text{ (Ax. } name) \qquad \text{where } \phi \text{ is a nonlogical axiom.}$$

$$\frac{\phi, \Gamma \vdash \Delta}{\Gamma \vdash \Delta} \text{ (Thm. } name) \qquad \text{where } \phi \text{ is an previously proved theorem.}$$

So, for example, to introduce the empty set axiom.

$$\frac{\forall x. x \notin \emptyset \vdash \emptyset \subseteq A}{\vdash \emptyset \subseteq A} \text{ (Ax. EmptySet)}$$

Assignment: Prove the following sequents.

1. $\vdash \quad \forall A.\ A \subseteq \emptyset \Rightarrow A = \emptyset$
2. $\vdash \quad \forall A.\ (\exists x. x \in A) \Rightarrow \neg(A = \emptyset)$