HW 13 **Prof. Caldwell**
Due: 13 October 2009 **COSC 2300**

Here are definitions of equality for sets, subset and the union and intersection and union operations:

$$\begin{aligned}
x \notin A &\stackrel{\text{def}}{=} \neg(x \in A)\\
A \subseteq B &\stackrel{\text{def}}{=} \forall x. x \in A \Rightarrow x \in B\\
A = B &\stackrel{\text{def}}{=} \forall x. x \in A \Leftrightarrow x \in B\\
x \in (A \cap B) &\stackrel{\text{def}}{=} x \in A \wedge x \in B\\
x \in (A \cup B) &\stackrel{\text{def}}{=} x \in A \vee x \in B
\end{aligned}$$

The following two rules allow definitions to be unfolded in a sequent.

$$\frac{\Gamma_1, \psi, \Gamma_2 \vdash \Delta}{\Gamma_1, \phi, \Gamma_2 \vdash \Delta}\ (\text{def of } op) \qquad \text{where } \phi \stackrel{\text{def}}{=} \psi \text{ is a definition of } op.$$

$$\frac{\Gamma \vdash \Delta_1, \psi, \Delta_2}{\Gamma \vdash \Delta_1, \phi, \Delta 2}\ (\text{def of } op) \qquad \text{where } \phi \stackrel{\text{def}}{=} \psi \text{ is a definition of } op.$$

So, an instance of the definition rule to unfold $\subseteq$ would appear as follows:

$$\frac{\forall x. x \notin \emptyset \vdash \forall x. (x \in \emptyset \Rightarrow x \in A)}{\forall x. x \notin \emptyset \vdash \emptyset \subseteq A}\ (\text{def of } \subseteq)$$

Also, for any nonlogical axiom or previously proved theorem ϕ referred to by the name *name*, the following rules holds.

$$\frac{\phi, \Gamma \vdash \Delta}{\Gamma \vdash \Delta}\ (\text{Ax. } name) \qquad \text{where } \phi \text{ is a nonlogical axiom.}$$

$$\frac{\phi, \Gamma \vdash \Delta}{\Gamma \vdash \Delta}\ (\text{Thm. } name) \qquad \text{where } \phi \text{ is an previously proved theorem.}$$

So, for example, to introduce the empty set axiom.

$$\frac{\forall x. x \notin \emptyset \vdash \emptyset \subseteq A}{\vdash \emptyset \subseteq A}\ (\text{Ax. EmptySet})$$

Assignment: Prove the following sequents.

1. $\vdash \ \forall A. (A \cap \emptyset) = \emptyset$
2. $\vdash \ \forall A. \forall B. (A \cap B) \subseteq A$
3. $\vdash \ \forall A. \forall B.\ A \subseteq B \Rightarrow (A \cap B) = A$
4. $\vdash \ \forall A. \forall B.\ ((A \cup B) = B) \Rightarrow A \subseteq B$