HW 17 **Prof. Caldwell**
Due: 3 November 2009 **COSC 2300**

This one may be more difficult.
Recall the definition of a closure.

Definition 0.1. Given a relation $R \subseteq A \times A$ and a property P of the relation, the *closure* of R with respect to P is the relation S such that $P(S)$ and $R \subseteq S$ and S is the smallest such relation,

$$\begin{array}{l} closure(R,P) = S \\ \quad \text{iff } (P(S) \wedge R \subseteq S) \wedge \forall T : T \subseteq A \times A \Rightarrow ((R \subseteq T \wedge P(T)) \Rightarrow S \subseteq T) \end{array}$$

Definition 0.2. The *diagonal relation* over a set A is the relation

$$\Delta_A = \{\langle x, y\rangle \in A \times A | x = y\}$$

Definition 0.3. The predicate $Refl_A(R)$ means R is reflexive with respect to the set A.

$$Refl_A(R) \stackrel{\text{def}}{=} \forall x : A.xRx$$

We proved the following in notes:

Theorem 0.1. If $R \subseteq A \times A$ then the reflexive closure of R is the relation $R \cup \Delta_A$

Proof: The theorem says

$$closure(R, Refl_A) = R \cup \Delta_A$$

Thus, to show that the $R \cup \Delta_A$ is the reflexive closure, (by the definition of *closure*) we must show three things:

$$\begin{array}{ll} i.) & Refl_A(R \cup \Delta_A) \\ ii.) & R \subseteq (R \cup \Delta_A) \\ iii.) & \forall T : T \subseteq A \times A \Rightarrow ((Refl_A(T) \wedge R \subseteq T) \Rightarrow (R \cup \Delta_A) \subseteq T) \end{array}$$

i.) We must show that $\forall x : A.\langle x, x\rangle \in (R \cup \Delta_A)$. Choose an arbitrary $x \in A$. Then, by the membership property of unions, we must show that $\langle x, x\rangle \in R$ or $\langle x, x\rangle \in \Delta_A$. But by the definition of membership in a comprehension, $\langle x, x\rangle \in \Delta_A$ iff $\langle x, x\rangle \in A \times A$ (which is obviously true) and if $x = x$. So, we conclude that (i) holds.

ii.) We must show that $R \subseteq (R \cup \Delta_A)$. But this is true by Thm 5.8 from Chapter 5.
iii.) Finally, we must show that $R \cup \Delta_A$ is the least such set, *i.e.* that

$$\forall T : T \subseteq A \times A \Rightarrow ((R \subseteq T \wedge Refl_A(T)) \Rightarrow (R \cup \Delta_A) \subseteq T)$$

To see this, choose an arbitrary relation $T \subseteq A \times A$. Assume $R \subseteq T$ and $Refl_A(T)$. We must show that $(R \cup \Delta_A) \subseteq T$. Let x be an arbitrary element of $(R \cup \Delta_A)$. Then, there are two cases: $x \in R$ or $x \in \Delta_A$. If $x \in R$, since we have assumed $R \subseteq T$, we know $x \in T$. In the other case, $x \in \Delta_A$, that is, x is of the form $\langle y, y\rangle$ for some y in A. But since we assumed $Refl_A(T)$, we know that $\forall z : A.\langle z, z\rangle \in T$ so, in particular, $\langle y, y\rangle \in T$, *i.e.* $x \in T$.

□

Definition 0.4. Now, if $R \subseteq A \times B$ then the *inverse* of R is

$$R^{-1} \stackrel{\text{def}}{=} \{\langle y, x \rangle \in B \times A \mid \langle x, y \rangle \in R\}$$

Definition 0.5. The predicate $Sym(R)$ means $R \subseteq A \times A$ is symmetric.

$$Sym(R) \stackrel{\text{def}}{=} \forall x, y \colon A.\ xRy \Rightarrow yRx$$

Assignment:
1.) Using the proof given above as a model, prove the following theorem.

Theorem 0.2. $closure(R, Sym) = R \cup R^{-1}$