Area: Computer Theory

Three questions each from three parts

Answer five of the nine questions

Part 1. Foundations of Computing

Let f be the function recursively defined on the nonnegative integers by

$$f(k,n) \overset{\text{def}}{=} \begin{array}{l} \text{if} \quad k = 0 \quad \text{then} \quad n+1 \\ \text{else if} \quad n = 0 \quad \text{then} \quad f(k-1,1) \\ \text{else} \quad f(k-1, f(k, n-1)) \end{array}$$

It is clear that for every nonnegative integer n, $f(0,n) = n+1$.
Use *mathematical induction* to prove for every nonnegative integer n, $f(1,n) = n+2$.

Show that if the following function for computing `n`! halts, then it is correct. Clearly state and prove the conditional theorems required to show (partial) correctness.

```
function fac(n:nonneg int):positive int
  f:positive int
  i:positive int





  f := 1
  i := 2

  {(f = (i-1)! and i <= n+1) or (f = 1 and i > n+1 and n = 0)}

  while  i <= n  do





      f := f * i





      i := i + 1





  end while





  return(f)
```

Let L be the set of all nonempty strings over $\{a, b\}$ where the a's and b's alternate. Assume the strings a and b are in L,

- Give an inductive definition for L. You may use the string operations append on the left, `head`, `tail`, and = (either to test for the empty string or to compare string elements). Be sure to apply `head` and `tail` only to nonempty strings.
- Give a regular expression that matches exactly the elements of L.

Part 2. Theory of Computation

Let f be the function recursively defined on the nonnegative integers by

$$f(k,n) \stackrel{\text{def}}{=} \begin{array}{l} \text{if} \quad k = 0 \quad \text{then} \quad n+1 \\ \text{else if} \quad n = 0 \quad \text{then} \quad f(k-1,1) \\ \text{else} \quad f(k-1, f(k, n-1)) \end{array}$$

Use the *Universality, Parameter, and Fixed Point Theorems* (see Davis, Sigal, and Weyuker, chapter 4) to prove there is a partially computable function f satisfying the given recursive definition.

Let F and G be the functions recursively defined on the nonnegative integers by

$$\begin{aligned} F(0) &= 1 \\ G(0) &= 1 \\ F(n+1) &= F(n) + G(n) \\ G(n+1) &= F(n) \cdot G(n) \end{aligned}$$

Prove F and G are primitive recursive functions.

Use *Rice's Theorem* (see Davis, Sigal, and Weyuker, chapter 4) to show none of the following sets is recursive.

$$
\begin{aligned}
A &= \{n \in N \mid (\forall x \in N)[\Phi(x,n)\uparrow]\} \\
B &= \{n \in N \mid (\exists x \in N)[\Phi(x,n) > x^2]\} \\
C &= \{n \in N \mid \Phi(x,n) \text{ is defined for all but finitely many x}\}
\end{aligned}
$$

Here N is the set of all nonnegative integers.