

Teaching Natural Deduction as a Subversive Activity

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Abstract

In this paper we argue that sequent proofs systems are a better deductive apparatus for introducing proofs to students than are natural deduction systems. Natural deduction derivations of superintuitionistic theorems (theorems of classical logic that are not intuitionistically valid) require indirect and often obtuse constructions. Constructing derivations in sequent systems is uniform, driven by syntax. The strong claim often made for natural deduction is that derivations closely reflect the logical thought process. We argue here that, for the superintuitionistic theorems, classical sequent derivations also reflect a logical thought process which is more concise and more direct than their natural deduction counterparts.

1 Introduction

In 1935 Gerhard Gentzen introduced natural deduction and sequent proof systems for both intuitionistic and classical logic [3]. Pedagogically, natural deduction has been preferred over sequent calculi. There are a large number of introductory logic texts based on natural deduction [4, 6, 12] while there are very few that use sequent systems. The recently published [10] is an exception as is the author's own [2] (unpublished).

We offer the following anecdote as motivation for this paper.

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Some years ago, teaching computer science students natural deduction proofs from a well known text, I assigned a number of exercises. One of the problems was to give a natural deduction derivation for the following theorem of classical logic: $(\phi \wedge \psi) \Rightarrow \sigma \Rightarrow ((\phi \Rightarrow \sigma) \vee (\psi \Rightarrow \sigma))$. At the following lecture, students were very agitated. No one in the class had managed to prove the formula and a number of students had convinced themselves the theorem was not valid. I contacted the authors pointing out that difficulties students have proving superintuitionistic theorems in the natural deduction system do not arise when using a sequent system. One of the authors replied, “We like it this way.”

One interpretation of the author’s reply is that, raising questions about the validity of classical logic is one of the goals of using natural deduction; a subversive teaching strategy indeed [8]. Questions about classical logic are good ones, but is this the best way to teach beginning students classical logic itself?

1.1 Argument

Our argument for using a sequent calculus for introducing students to proof in classical logic¹ is based on two claims:

- i.) The method of constructing sequent derivations is uniform.
- ii.) For superintuitionistic theorems the explanations embedded in sequent derivations are better than those contained in natural deduction derivations of the same theorem.

Uniform Derivations It is well known that sequent systems for classical propositional logic² provide a framework for uniform proof search [7, 11]. Finding a sequent derivation is a syntax directed affair which proceeds by structurally decomposing the formulas in the sequent to be proved; finding a natural deduction proof can require significant ingenuity³ on the part of the student.

Explanatory Content A *strong claim* often made for natural deduction is that derivations somehow reflect the process of *logical reasoning*; the way experts think when constructing a proof. Gentzen himself claimed this for his intuitionistic natural deduction system *NJ*. He wrote [3, pp.80]: “5.11 [An advantage of *NJ* is] A close affinity to actual reasoning...”. Gentzen did not make the same claim for his classical natural deduction system *NK*. We take this strong claim

¹We have limited our discussion here to propositional logic but sequent systems work equally well for introducing first order logic.

²Not all formulations of the sequent rules have this property, see *e.g.* [7, pp.16], shared contexts (sometimes called multiplicative) provide for easy search.

³More than one proponent of natural deduction has commented that maybe propositional logic shouldn’t be easy or that perhaps it isn’t, really. This too would seem to be a subversive notion. It might be compared to the idea that because arithmetic using Roman numerals is difficult, that perhaps arithmetic just is difficult.

to be the reason proponents of natural deduction prefer it. Here, we argue that sequent derivations, though easier to construct, encode similarly cogent proofs.

It is perhaps not inconceivable that the claimed correspondence between the logical thought process and the resulting natural deduction derivation is only available to the author of the derivation, and not the reader, though this seems unlikely. If not, the structure of the derivation itself in some way records the logical thought process and this content is available to authors and readers alike. If so, natural deduction derivations somehow encode an account of the logical reasoning process and it should be possible to reconstruct an informal proof from the formal derivation that reflects this structure. Of course it is. This last claim is also true for sequent derivations⁴

Our method of arguing that sequent derivations contain a similar record of logical reasoning will be to compare the natural language proofs extracted from derivations of the same theorem in both systems. We claim that the proofs extracted from sequent derivations of superintuitionistic are more perspicuous than those extracted from natural deduction derivations; of course the reader must decide this for herself. Since we agree that for theorems of intuitionistic logic, natural deduction derivations are natural, we will examine derivations of superintuitionistic theorems.

2 Natural Deduction

Here, we use Prawitz[9] style tree structured derivations with explicit labels for assumptions. Linear representations exist as well [6, 4]. The proof rules for Natural Deduction are presented in Fig. 1. The fundamental idea of the natural deduction systems is that each logical connective is represented by two rules: one describing how the connective is to be *introduced* into a proof and the other describing how a connective is *eliminated* or decomposed.

All of the rules⁵ except *reductio ad absurdum* (RAA) are intuitionistically acceptable [9]. We can classify theorems by their proofs. Theorems having at least one proof that does not use the rule (RAA) are *intuitionistic*. Theorems for which the use of RAA is essential are *superintuitionistic*. There is no shortage of such theorems. Note that the rule RAA includes a negation in the premises of the rule while no explicit negation appears in the conclusion, this gives it the nature of an elimination rule. But, since superintuitionistic theorems (even ones containing no occurrence of the negation (\neg) symbol) require the use of this rule, derivations of such theorems require the introduction of a negated formula. This is one sense in which the resulting proofs are *unnatural*. Classical natural deduction does not have the subformula property⁶ [5].

⁴In [2] an recursive algorithm for translating sequent proofs into natural language is presented. A similar algorithm for natural deduction proofs could similarly be designed.

⁵Prawitz used *reductio ad absurdum* (RAA) as his classical rule. Gentzen [3] extend his intuitionistic system *NJ* to classical logic by adding excluded middle (which we prove below using RAA). Adding a rule for double negation elimination will do as well.

⁶The subformula property (which *does hold* for cut-free derivations in the sequent calculus) says that all formulas occurring in a derivation are subformulas of the goal. Also, formulations

INTRODUCTIONS	ELIMINATIONS
$\frac{\phi \quad \psi}{\phi \wedge \psi} \wedge i$	$\frac{\phi \wedge \psi}{\phi} \wedge e_1 \quad \frac{\phi \wedge \psi}{\psi} \wedge e_2$
$\frac{\phi}{\phi \vee \psi} \vee i_1 \quad \frac{\psi}{\phi \vee \psi} \vee i_2$	$\frac{\begin{array}{c} [\phi]^j \\ \vdots \\ \phi \vee \psi \end{array} \quad \begin{array}{c} [\psi]^k \\ \vdots \\ \sigma \end{array}}{\sigma} \vee e$
$\frac{\begin{array}{c} [\phi]^j \\ \vdots \\ \psi \end{array}}{\phi \Rightarrow \psi} \Rightarrow i$	$\frac{\phi \quad \phi \Rightarrow \psi}{\psi} \Rightarrow e$
$\frac{\begin{array}{c} [\phi]^j \\ \vdots \\ \perp \end{array}}{\neg \phi} \neg i$	$\frac{\phi \quad \neg \phi}{\perp} \neg e$
$\frac{\perp}{\phi} \text{RAA}$	$\frac{\begin{array}{c} [\neg \phi]^j \\ \vdots \\ \perp \end{array}}{\phi} \perp e$

Figure 1: Natural Deduction Proof Rules

3 Sequent Calculus

A *sequent* is a pair of collections (possibly empty) of formulas (separated here the symbol \vdash) $\Gamma \vdash \Delta$. The collection Γ is called the *antecedent* and Δ is the *succedent*. A sequent characterizes the state of a derivation. The formulas in the antecedent (on the left side of \vdash), are the open assumptions. The right formulas, in the succedent, are the open goals, any one of which may be shown. By a denotational interpretation [7, pp.47] of multisuccedent sequents, $\Gamma \vdash \Delta$ is valid if the conjunction of the formulas in the antecedent implies the disjunction of the formulas in the succedent.

The sequent calculus proof rules are shown in Fig. 2. In this formulation of the sequent rules, Γ and Δ are sequences (lists) of formulas⁷. In the sequent calculus, there are two rules for each connective, one that shows, based on the principle connective, how to decompose an open assumption and another showing how to decompose a connective occurring in an open goal. There are two axiom rules.

Note that there is no equivalent of RAA in the sequent rules, though it is derivable. We start the derivation with the assumption⁸ $\neg \phi \Rightarrow \perp$ and show ϕ .

⁷We have found that for beginning students, lists are more concrete than sets or multisets: the list specification allows us to easily specify that the formula operated upon is dropped from the antecedent or succedent above.

⁸Proponents sometimes claim that an advantage of natural deduction is that it allows

$$\begin{array}{c}
\text{AXIOMS} \\
\frac{}{\Gamma_1, \phi, \Gamma_2 \vdash \Delta_1, \phi, \Delta} (\text{Ax}) \quad \frac{}{\Gamma_1, \perp, \Gamma_2 \vdash \Delta} (\perp \text{Ax}) \\
\\
\text{LEFT RULES} \qquad \qquad \qquad \text{RIGHT RULES} \\
\frac{\Gamma_1, \phi, \psi, \Gamma_2 \vdash \Delta}{\Gamma_1, (\phi \wedge \psi), \Gamma_2 \vdash \Delta} (\wedge \text{L}) \quad \frac{\Gamma \vdash \Delta_1, \phi, \Delta_2 \quad \Gamma \vdash \Delta_1, \psi, \Delta_2}{\Gamma \vdash \Delta_1, (\phi \wedge \psi), \Delta_2} (\wedge \text{R}) \\
\frac{\Gamma_1, \phi, \Gamma_2 \vdash \Delta \quad \Gamma_1, \psi, \Gamma_2 \vdash \Delta}{\Gamma_1, (\phi \vee \psi), \Gamma_2 \vdash \Delta} (\vee \text{L}) \quad \frac{\Gamma \vdash \Delta_1, \phi, \psi, \Delta_2}{\Gamma \vdash \Delta_1, (\phi \vee \psi), \Delta_2} (\vee \text{R}) \\
\frac{\Gamma_1, \Gamma_2 \vdash \phi, \Delta \quad \Gamma_1, \psi, \Gamma_2 \vdash \Delta}{\Gamma_1, (\phi \Rightarrow \psi), \Gamma_2 \vdash \Delta} (\Rightarrow \text{L}) \quad \frac{\Gamma, \phi \vdash \Delta_1, \psi, \Delta_2}{\Gamma \vdash \Delta_1, (\phi \Rightarrow \psi), \Delta_2} (\Rightarrow \text{R}) \\
\frac{\Gamma_1, \Gamma_2 \vdash \phi, \Delta}{\Gamma_1, \neg \phi, \Gamma_2 \vdash \Delta} (\neg \text{L}) \quad \frac{\Gamma, \phi \vdash \Delta_1, \Delta_2}{\Gamma \vdash \Delta_1, \neg \phi, \Delta_2} (\neg \text{R})
\end{array}$$

Figure 2: Sequent Proof Rules

$$\frac{\frac{\frac{}{\phi \vdash \phi} \text{Ax}}{\vdash \neg \phi, \phi} \neg R \quad \frac{}{\perp \vdash \phi} \perp \text{Ax}}{\neg \phi \Rightarrow \perp \vdash \phi} \Rightarrow L$$

Figure 3: Sequent derivation of RAA

A natural language translation of the sequent derivation follows:

Since we have assumed $(\neg \phi \Rightarrow \perp)$, if we show that $\neg \phi$ or ϕ holds, we may assume \perp to show ϕ . To see that $\neg \phi$ holds, we assume ϕ . We are left to show that ϕ holds but we have assumed it so this case is trivial. Having discharged $\neg \phi$, we are free to assume \perp and must show ϕ , but this is trivial because we have assumed false. Q.E.D.

This derivation illustrates how the sequent calculus rules account for classical reasoning. More than one formula may appear in the succedent of a sequent and any one of the formulas there may be operated on by a right rule. This happens in the left branch of the derivation in Fig. 3 after the application of the $\Rightarrow L$ rule. For classical logic the disjunction property⁹ does not hold. Critics of sequent calculus sometimes complain that having more than one goal in the succedent is problematic but, in a rather precise way, this is what it means to be superintuitionistic *i.e* more than one formula is required in the succedent¹⁰ to

for reasoning from assumptions. In the sequent system, assumptions may be added to the antecedent of the initial goal.

⁹The disjunction property is, if $\Gamma \vdash \phi \vee \psi$ is provable, then $\Gamma \vdash \phi$ is provable or $\Gamma \vdash \psi$ is provable.

¹⁰The intuitionistic sequent calculus is obtained from the rules in Fig. 2 by restricting the

complete the proof. The $\vee R$ rule justifies multiple goals perfectly well: to prove $\phi \vee \psi$, it is enough to prove either ϕ or ψ . The power of classical sequent calculus comes from changing the focus from one formula on the right to another. In the proof given above of RAA, the application of the $\neg R$ rule is focused on the formula $\neg\phi$, in the next step it turns out that ϕ is the formula proved. It is impossible to say which of ϕ or $\neg\phi$ has been proved because neither has. This is the nature of sequent derivations of superintuitionistic theorems.

4 Comparative Analysis

In this section we compare the two systems by considering both natural deduction derivations and sequent derivations of the following two formulas.

$$[XM] \quad \phi \vee \neg\phi \qquad [GDM] \quad (\phi \wedge \psi) \Rightarrow \sigma \Rightarrow ((\phi \Rightarrow \sigma) \vee (\psi \Rightarrow \sigma))$$

These formulas are superintuitionistic tautologies of classical logic. The formula XM is the *law of excluded middle*. The formula GDM is the one students struggled so with in the motivating anecdote. It is a generalization of one direction of De Morgan's law for pushing a negation down over a conjunction $\neg(\phi \wedge \psi) \Rightarrow (\neg\phi \vee \neg\psi)$. To see this, consider the case where σ is \perp and note that $(\phi \Rightarrow \perp) \Leftrightarrow \neg\phi$. Readers may suspect that we have carefully chosen these examples to support our point, not so, any number of other important superintuitionistic theorems for this analysis.

Law of Excluded Middle A natural deduction derivation and a sequent derivation of excluded middle are shown in Fig. 4.

$$\begin{array}{c}
 \frac{[\phi]}{\phi \vee \neg\phi} \vee i_1 \qquad \frac{[\neg(\phi \vee \neg\phi)]}{\neg\phi} \neg e \\
 \frac{\frac{\frac{\perp}{\neg\phi} \neg i}{\phi \vee \neg\phi} \vee i_2 \qquad [\neg(\phi \vee \neg\phi)] \neg e}{\frac{\perp}{\phi \vee \neg\phi} \text{RAA}} \neg e \qquad \frac{\frac{\frac{\perp}{\phi \vdash \phi} \text{Ax}}{\vdash \phi, \neg\phi} \neg r}{\vdash \phi \vee \neg\phi} \vee r
 \end{array}$$

Figure 4: Natural Deduction and Sequent derivations of XM

Perhaps the first thing to note about these derivations is how complex the natural deduction derivation is in relation to the sequent derivation; there are more symbols (it takes more ink), it takes more space on the page and the shape of the natural deduction derivation is more complex than the sequent derivation of the same formula. Also note that at every step of the sequent derivation, the rule to apply is determined by the syntax of the formulas in the sequent. In general, if more than one rule applies, it does not matter which one is applied

succedent to contain at most one formula.

A natural deduction derivation of GDM is shown in Fig. 5. An informal account of the argument in the derivation is given as follows:

To prove GDM assume the antecedent $((\phi \wedge \psi) \Rightarrow \sigma)$ and then to show the consequent $((\phi \Rightarrow \sigma) \vee (\psi \Rightarrow \sigma))$ (call this formula C .)^a. We assume $\neg C$ and derive a contradiction^b. To see how, assume ϕ and assume ψ to get $\phi \wedge \psi$. We use this formula to eliminate the implication in the first assumption $(\phi \wedge \psi) \Rightarrow \sigma$, this yields σ . We discharge the assumption ϕ by introducing the implication $\phi \Rightarrow \sigma$. We use $\forall i$ to get C again. An application of $\neg E$ gives \perp ^c. Use $\perp E$ to get σ again and now introduce the implication $\psi \Rightarrow \sigma$ discharging the assumption ψ . Another application of $\forall I$ yields C again. With another copy of the assumption $\neg C$ we derive \perp . We apply RAA, discharging the assumption $\neg C$ to get C . Implication introduction discharges the first assumption and completes the proof.

^aAt this point, one might try to prove C directly which can not be done from the assumption.

^bWe have encountered a seeming circularity again, we started trying to prove C and now, as a subgoal of the strategy for using RAA must prove C . The experienced prover pushes on.

^cThe assumption ψ and $\neg C$ are still open so we go around again.

Consider the sequent derivation of GDM shown in Fig. 6

$$\begin{array}{c}
\frac{}{\phi, \psi \vdash \phi, \sigma} \text{Ax} \quad \frac{}{\phi, \psi \vdash \psi, \sigma} \text{Ax} \\
\frac{}{\phi, \psi \vdash \phi \wedge \psi, \sigma} \wedge r \\
\frac{}{\phi \vdash \phi \wedge \psi, \psi \Rightarrow \sigma} \Rightarrow r \quad \frac{}{\sigma, \phi \vdash \sigma, \psi \Rightarrow \sigma} \text{Ax} \\
\frac{}{(\phi \wedge \psi) \Rightarrow \sigma, \phi \vdash \sigma, \sigma \Rightarrow \psi} \Rightarrow r \\
\frac{}{(\phi \wedge \psi) \Rightarrow \sigma \vdash \phi \Rightarrow \sigma, \psi \Rightarrow \sigma} \vee r \\
\frac{}{(\phi \wedge \psi) \Rightarrow \sigma \vdash (\phi \Rightarrow \sigma) \vee (\psi \Rightarrow \sigma)} \Rightarrow r \\
\frac{}{\vdash ((\phi \wedge \psi) \Rightarrow \sigma) \Rightarrow ((\phi \Rightarrow \sigma) \vee (\psi \Rightarrow \sigma))} \Rightarrow r
\end{array}$$

Figure 6: Sequent derivation of GDM

An account of this proof is as follows:

To prove GDM assume the antecedent $((\phi \wedge \psi) \Rightarrow \sigma)$ and show $((\phi \Rightarrow \sigma) \vee (\psi \Rightarrow \sigma))$. Since the goal is a disjunction, it is enough to show one of $(\phi \Rightarrow \sigma)$ or $(\psi \Rightarrow \sigma)$. To show $\phi \Rightarrow \sigma$, assume ϕ and show one of σ or $(\psi \Rightarrow \sigma)$. Since we have assumed $((\phi \wedge \psi) \Rightarrow \sigma)$, if we show $(\phi \wedge \psi)$ we can assume σ to show σ or $(\phi \Rightarrow \sigma)$, but this is trivial. To see that $(\phi \wedge \psi)$ holds: assume ψ and show that one of $\phi \wedge \psi$ or ψ holds. To show $\phi \wedge \psi$ we must show ϕ and we must show ψ . But we have previously assumed both of these formulas so the proofs are trivial. Q.E.D.

Remarks on the Derivations and the Proofs For superintuitionistic theorems, natural deduction derivations *must* take a circuitous path; deriving a contradiction from a negated assumption. The goal of deriving a contradiction often leads to the introduction of more assumptions which must in turn

themselves be discharged. Natural deduction derivations require a sophisticated combination of forward and backward reasoning.

Sequent derivations are constructed strictly by working backwards and are driven by the syntactic structure of the formulas in the sequent. Natural language proofs extracted from sequent derivations are more direct and are at least as readable, if not more so, than their natural language counterparts.

5 Other objections to Sequent Calculus

Aside from the claims made *for* natural deduction, there are other objections made against sequent proof systems which we only briefly address here.

Sequent derivations require more writing. A prominent complaint is that sequent derivations require more writing. It certainly is that case that there is more *copying* of formulas in sequent derivations than in natural deduction derivations. This is evident by examining the rules. It not the case that sequent derivations always require more ink or more paper. In fact, in both of the examples presented here, the sequent derivations require fewer symbols and take less paper. It seems that the advantages of the sequent systems outweighs this complaint.

Natural deduction rules are more natural. The claim is sometimes made that the form of the natural deduction rules introduction and elimination rules is somehow more intuitive than the left and right rules of the sequent calculus. It is well known [7, pp.14-15] that elimination rules correspond to left rules and introduction rules correspond to right rules. Justification for the $\forall R$ rule seems to be critical.

Tableaux systems are better than sequents. Tableaux systems are quite similar to sequent calculi and do require less copying of formulas. Tableaux provide a calculational mechanism (like sequent calculi and truth tables) for determining validity but an (unjustified) claim is that their explanatory content is not as direct as that provided by a sequent derivation.

6 Conclusions

Failure to find a natural deduction proof has led more than one student to reject the classically valid theorems as absurd. On the one hand, this reaction can be dismissed as obviously flawed and unreasonable¹¹; though reasonable people share these views. The sequent calculus has the advantage that finding a derivation is a mechanical process. We have also claimed that the arguments embedded in them are, in the superintuitionistic case better than the rather convoluted reasoning behind corresponding natural deduction derivations.

¹¹This dismissal is what Anderson and Belknap [1, pp.4] call *The official view i.e.* that the misunderstanding is naive and can be cleared up by a truth table.

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