## Proof extraction from multi-succedent intuitionistic derivations

Roy Dyckhoff (University of St Andrews)

James Caldwell (University of St Andrews; University of Wyoming)

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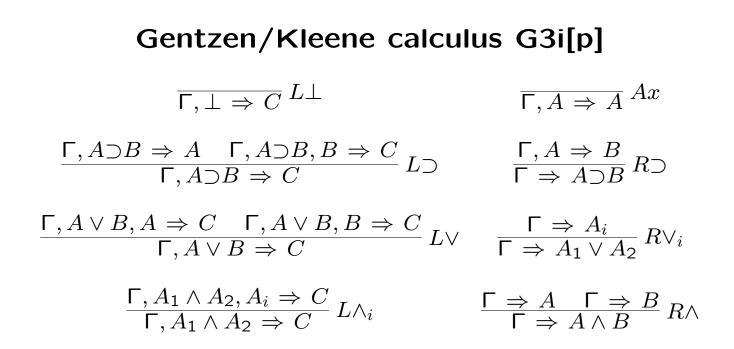
#### Outline of talk

- 1. Proof Search in Type Theory (NuPRL)
- 2. Sequent Calculi: Gentzen/Kleene G3i; Maehara/Kleene G3im
- 3. Complexity issue (Egly-Schmitt)
- 4. Translation from **G3i** to **G3im**
- 5. Translation from G3im to G3i
  - (a) Using Cuts
  - (b) Schmitt-Kreitz translation
  - (c) Mints translation
  - (d) Egly-Schmitt translation
- 6. Translations from G3im to G3i + CUT
- 7. Cut-elimination in G3i+CUT
- 8. Conjecture and Issues
- 9. Experiments

# Proof Search in Type Theory (NUPRL)

Proof search in the NUPRL proof assistant is based on an automated theorem prover, using (in effect) a multi-succedent intuitionistic calculus; from this algorithms (i.e. ordinary lambda terms) are extracted using a translation.

Multi-succedent calculus is preferred for various reasons: use of efficient matrix/connection methods (Bibel, Wallen) and label unification (Otten); complexity issue (see below); perhaps historical reasons. From the point of view of provability, all these calculi (to follow) are equivalent. From the perspective of finding algorithms, whether they are or are not is unclear.



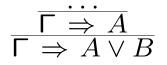
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## Complexity (Egly-Schmitt)

Egly and Schmitt [1] showed that there is a sequence  $(S_n)$  of intuitionistic sequents s.t., for each n,  $S_n$  is derivable in **G3im** (using a derivation with 6n - 2 leaves) but that every derivation of  $S_n$  in **G3i** has at least  $2^n$  leaves. Thus, **G3i** cannot polynomially simulate **G3im**. (The converse is known, that **G3im** can polynomially simulate **G3i**.)

#### Translation from G3i into G3im

Straightforward use of Weakening on the right, e.g. a step



becomes

$$\frac{\overline{\Gamma \Rightarrow A}}{\Gamma \Rightarrow A, B} RW$$
$$\overline{\Gamma \Rightarrow A \lor B}$$

from which uses of RW can be eliminated (by pushing it up to leaves or to  $R \supset$  and  $R \forall$  inferences). This elimination adds formulae to the RHS, hence the multiple succedents.

## Translation from G3im into G3i

- 1. Via G3i+CUT (e.g. Cuts distribute & over  $\lor$ )
- 2. Schmitt & Kreitz (use of modified  $\mathbf{G3i}$ , with more complex initial sequents)
- 3. Mints (permutation argument)
- 4. Egly & Schmitt (another permutation argument)

The last three are complex, hard to analyse w.r.t. relationship with natural deduction proofs. We therefore consider only the first.

#### Translation from G3im to G3i + CUT

Wherever  $\Delta$  is a multiset, let  $\delta$  be a disjunction of all the elements in  $\Delta$ . It is then routine to show that if  $\Gamma \Rightarrow \Delta$  is derivable in **G3im** then  $\Gamma \Rightarrow \delta$  is derivable in **G3i**. Suppose for example the final step is by  $R \land$ , giving  $\Gamma \Rightarrow A \land B, \Delta'$ . A  $R \land$  step on the transforms of the derivations of the premisses followed by a cut with  $(A \lor \delta') \land (B \lor \delta') \Rightarrow (A \land B) \lor \delta'$  gives us  $\Gamma \Rightarrow (A \land B) \lor \delta'$ .

There are alas tedious issues concerning the order in which the elements of a multiset are disjoined. Some easy simplifications are possible (e.g. doing two cuts before rather than one step after the  $R \wedge$  step).

## Cut-elimination in G3i+CUT

We now have, starting from a G3im derivation, a derivation in G3i+CUT. Our goal is to get a natural deduction (represented as a lambda term). Use of a lambda notation for derivations is, once we are in G3i+CUT, routine. We can now go by several routes: translate directly to ordinary lambda terms and normalise, or eliminate cuts and translate to normal lambda terms.

We chose (for reasons now obscure) the second route, although there are *caveats* (see e.g. [2,3]):

- 1. Not all cut elimination systems in G3i+CUT are SN;
- 2. Not all cut elimination systems in **G3i+CUT** are confluent;
- 3. Not all cut elimination systems in G3i+CUT simulate beta-reduction.

# Cut-elimination in G3i+CUT, 2

[We chose the second route], using a particular SN system of cut reduction rules. The lack of confluence shouldn't be a problem, provided that the possible cut-free terms one obtains all have the same meaning as (normal) lambda terms; likewise, the failure to simulate beta-reduction isn't a problem, we are only interested in the end result, a normal lambda term.

In part, checking the first of these provisos means checking, in lambda calculus, lots of equations like

[w/x][xM/y]N = [w[w/x]M/y][w/x]N

corresponding to cut-reduction steps.

#### Conjecture and issues

We conjecture that, whenever a G3i derivation d is found for a sequent, its interpretation as a normal lambda term can also be obtained by interpreting (as above) at least one G3im derivation, namely the interpretation of d in G3im. Thus, that the interpretation (using cuts and cut-elimination) from G3im into the space of normal lambda terms is surjective.

We are part-way to establishing this. Whether this is true for the translations of Schmitt-Kreitz, Mints or Egly-Schmitt is, we guess, much harder to establish (except perhaps negatively).

Whether the interpretation is surjective if we consider only small **G3im** derivations is unclear; a complexity argument might decide it. By "small" is meant, for example, linear in the size of the end-sequent, thus allowing the small **G3im** derivations of the Egly-Schmitt examples.

# Experiments

Caldwell has implemented the above methods, and experimented; the results are not yet conclusive.

Of course, the correctness of the implementation also has to be verified....

## Comment

Whether **G3im** is appropriate for searching for algorithms is not clear; one of its advantages (if used root-first as a sequent calculus) is the invertibility of almost all rules, but then one immediately ignores lots of possible algorithms. That seems to be a separate issue. In any case, a Herbelin-style calculus (as in Lengrand's talk) seems more appropriate.

#### References

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- 6. Schmitt, Kreitz. On transforming intuitionistic matrix proofs into standard sequent proofs, TABLEAUX workshop, 1995.