Teaching Natural Deduction as a Subversive Activity

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A Story

Lazy Professor: (To class) *Do exercise xxx from the text.*

The Problematic Exercise

Prove the following formula using Natural Deduction.

\[((P \land Q) \Rightarrow R) \Rightarrow ((P \Rightarrow R) \lor (Q \Rightarrow R))\]

... Two days pass.
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**The Problematic Exercise**

Prove the following formula using Natural Deduction.

\[
((P \land Q) \Rightarrow R) \Rightarrow ((P \Rightarrow R) \lor (Q \Rightarrow R))
\]

... Two days pass.

A Chorus of Students: *There is a typo in the book!*

Student A: *The formula is false.*

---

**Student A: Disputation**

If \( P \) and \( Q \) together obtain \( R \), then surely it is not always the case that either \( P \) alone or \( Q \) alone obtains \( R \) e.g. let \( P \) be *over 18* and \( Q \) be *male* and \( R \) be *must register for military service*\(^a\).

\(^a\)In the US women do not register with the selective service.
The story continues

Lazy Professor: *But it is valid.*

(Expressing sympathy and explaining that the counterexample, though compelling, is incorrectly formulated as stated; that predicates and quantifiers are required to formulate it and that in fact, when properly formulated, the obviously false thing is not valid.)

Student A: *Classical logic is clearly wrong.*

A Chorus of Students: *Yes, classical logic is wrong.*

... More time passes ...
The story continues

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(Expressing sympathy and explaining that the counterexample, though compelling, is incorrectly formulated as stated; that predicates and quantifiers are required to formulate it and that in fact, when properly formulated, the obviously false thing is not valid.)

Student A: *Classical logic is clearly wrong.*
A Chorus of Students: *Yes, classical logic is wrong.*

... More time passes ...

Lazy Professor: *Consider the student’s disputation of the following formula, is the student correct?*
Professor A: *It would seem so.*
Lazy Professor: *But the formula is valid*

After a moment of reflection ...

Professor A: *Ah, of course, there is no way to falsify it.*
Some Remarks and a Claim

- \((P \land Q) \Rightarrow R) \Rightarrow ((P \Rightarrow R) \lor (Q \Rightarrow R))\)
  - A so-called “paradox” of material implication.
  - Generalization of De Morgan (take \(R\) to be \(\bot\) and \(\neg \phi \equiv \phi \Rightarrow \bot\))
    \(\neg(P \land Q) \Rightarrow (\neg P \lor \neg Q)\)
  - It is a superintuitionistic theorem of classical logic (not intuitionistically provable) – such theorems are difficult to prove in natural deduction.
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A Claim

These difficulties do not arise when students are taught to do sequent proofs instead of Natural Deduction proofs.
A Note on “Subversive” Activities

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This is what Neil Postman\(^a\) called *Subversive Teaching* - the method leads students to raise interesting questions for themselves about accepted ideas.

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This is what Neil Postman\textsuperscript{a} called *Subversive Teaching* - the method leads students to raise interesting questions for themselves about accepted ideas.


But is it the best way to teach classical logic?
Intuitionist and Relevant logicians raise serious questions about classical logic, but is a student's first encounter with logic the best point to raise these questions?
Gentzen’s Proof Systems
Natural Deduction and Sequent Proof Systems

Natural Deduction and Sequent proof systems were introduced by Gentzen in 1935 in his paper *Investigations into Logical Deduction*\(^1\)

**Proof Systems**

\[ \mathcal{NJ} \] Intuitionistic Natural Deduction
\[ \mathcal{NK} \] Classical Natural Deduction
\[ \mathcal{LI} \] Intuitionistic Sequent Calculus
\[ \mathcal{LK} \] Classical Sequent Calculus

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Natural Deduction and Sequent Proof Systems

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### Proof Systems

- $\mathcal{NJ}$ Intuitionistic Natural Deduction
- $\mathcal{NK}$ Classical Natural Deduction
- $\mathcal{LJ}$ Intuitionistic Sequent Calculus
- $\mathcal{LK}$ Classical Sequent Calculus

### Relationships

- $\mathcal{NK}$ is obtained from $\mathcal{NJ}$ by adding a rule for *Tertium non datur* or *Reductio ad Absurdum*.
- Rather surprisingly, $\mathcal{LK}$ is obtained from $\mathcal{LJ}$ simply by allowing multiple formula on the right side.

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Natural Deduction Proof Rules

**Elimination Rules**

- $\frac{\phi \land \psi}{\phi} \land e_1$
- $\frac{\phi \land \psi}{\psi} \land e_2$

  - $\frac{\phi \lor \psi}{\sigma} \land e$
  - $\frac{\sigma \land e}{\sigma}$

- $\frac{\phi}{\phi \rightarrow \psi} \Rightarrow e$

- $\frac{\phi}{\neg \phi} \neg e$

- $\frac{\bot}{\phi} \bot e$

**Introduction Rules**

- $\frac{\phi}{\phi \land \psi} \land i$

  - $\frac{\phi}{\phi \lor \psi} \lor i_1$
  - $\frac{\psi}{\phi \lor \psi} \lor i_2$

  - $\frac{\phi \lor \psi}{\sigma} \lor e$
  - $\frac{\sigma \lor e}{\sigma}$

- $\frac{\phi \Rightarrow \psi}{\phi \Rightarrow \psi} \Rightarrow i$

  - $\frac{\phi}{\neg \phi} \neg \Rightarrow i$

- $\frac{\bot}{\neg \phi} \neg i$

**RAA**

- $\frac{\bot}{\phi}$
Comments on Natural Deduction

- There is an elegant symmetry in the Introduction and Elimination rules for the logical connectives ($\mathcal{N}$,$\mathcal{J}$).

- The Curry-Howard Isomorphism relates $\mathcal{N}$,$\mathcal{J}$ proofs with lambda terms.

- Any theorem of $\mathcal{N}$,$\mathcal{J}$ not provable in $\mathcal{N}$,$\mathcal{J}$ will require a use of $\text{Raa}$—such theorems are called superintuitionistic.

- $\text{Raa}$ breaks the symmetry of the intro/elim rules.

- There is a similarity between $\neg i$ and $\text{Raa}$ but the rule $\neg i$ is easily derived from $\Rightarrow i$ while $\text{Raa}$ is not derivable and introduces a negation.

- $\mathcal{N}$,$\mathcal{J}$ does not enjoy the subformula property—proofs of superintuitionistic theorems containing no negations will require the introduction of a negation.

- Failure to find a proof does not provide evidence one does not exist.
Comments on Natural Deduction

- There is an elegant symmetry in the Introduction and Elimination rules for the logical connectives (NJ).
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  - Propositions ⇔ Types

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Sequents

- A Sequent characterizes the state of a proof.

\[ \text{Velleman calls these Givens and Goals}\]

The semantics of a sequent is given by:

\[ [\Gamma \vdash \Delta] \text{ def } = (\bigwedge \phi \in \Gamma \phi) \Rightarrow \bigvee \psi \in \Delta \psi \]

Thus, a sequent is valid if some formula on the left is false or all formulas on the left are true and some formula on the right is as well.

LJ restricts \(|\Delta| \leq 1\) while LK has no restriction on the length of the succedent.

\(^2\text{Velleman, } \textit{How to Prove it: A Structured Approach, Cambridge Press 2006}\)
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- A sequent is a pair of (possibly empty) formula lists $\langle \Gamma, \Delta \rangle$
  - We write $\Gamma \vdash \Delta$.
  - $\Gamma$ is the antecedent.
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\end{align*}
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- \( \mathcal{LJ} \) restricts \(|\Delta| \leq 1 \) while \( \mathcal{LK} \) has no restriction on the length of the succedent.

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Sequent Proof Rules

**Axioms**

\[ \Gamma_1, \phi, \Gamma_2 \vdash \Delta_1, \phi, \Delta \] (Ax)

\[ \Gamma_1, \bot, \Gamma_2 \vdash \Delta \] (⊥Ax)

**Left Rules**

\[ \Gamma_1, \phi, \psi, \Gamma_2 \vdash \Delta \] (∧L)

\[ \Gamma_1, \phi, \psi, \Gamma_2 \vdash \Delta \] (∧R)

\[ \Gamma_1, \phi, \Gamma_2 \vdash \Delta \] (∨L)

\[ \Gamma, \phi \vdash \Delta_1, \psi, \Delta_2 \] (∨R)

\[ \Gamma_1, \psi, \Gamma_2 \vdash \Delta \] (⇒L)

\[ \Gamma, \phi \vdash \Delta_1, \psi, \Delta_2 \] (⇒R)

\[ \Gamma_1, \Gamma_2 \vdash \phi, \Delta \] (¬L)

\[ \Gamma, \phi \vdash \Delta_1, \Delta_2 \] (¬R)

**Right Rules**

\[ \Gamma_1, \psi, \Gamma_2 \vdash \Delta \] (¬R)

\[ \Gamma, \phi \vdash \Delta_1, \psi, \Delta_2 \] (¬R)
Remarks on Sequent Rules and Derivations

- Left rules correspond to Elimination rules and Right rules correspond to Introduction rules - there is no rule corresponding to $\text{Raa}$. 

Construction of sequent derivations is syntax driven.

1. Non-deterministically choose a compound formula in the left or right side and apply the corresponding rule.
2. If all formulas are atomic, check if the sequent is an instance of one of the axiom rules.
3. Repeat until all leaves of the tree are instances of axioms (and you have a proof) or until some atomic sequent turns out not to be an instance of an axiom rule (and you can build a counterexample.)

Failed derivations yield counterexamples.

- Consider an atomic sequent of the form $\Gamma \vdash \Delta$ that is not an instance of an axiom rule
- The assignment $(\lambda x. \text{if } (x \in \Delta) \text{ then } \text{True} \text{ else } \text{False})$ falsifies $\Gamma \vdash \Delta$
- It also falsifies any sequent rooting a derivation ending with $\Gamma \vdash \Delta$.

Proofs from assumptions are obtained by adding the assumed formulas to $\Gamma$. 

Caldwell (University of Wyoming) 
Teaching Natural Deduction ... 
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- Proofs from assumptions are obtained by adding the assumed formulas to \( \Gamma \).
Proof of Excluded Middle

- We will prove $\phi \lor \neg \phi$ in both $\mathcal{NK}$ and $\mathcal{LK}$.
- Examining the rules, it becomes clear that the explicit intro rules $\lor i_1$ and $\lor i_2$ can only arise from a proof of $\phi$ or a proof of $\neg \phi$ which is impossible without assumptions.
- The only rule that can help is $\text{RAA}$.

\[
\begin{array}{c}
\vdash (\phi \lor \neg \phi) \\
\vdash \bot [1] \\
\phi \lor \neg \phi \text{ RAA}
\end{array}
\]

- We must derive $\bot$ from the assumption $\neg (\phi \lor \neg \phi)$. 
The only rule having $\bot$ as its conclusion is $\neg e$.

\[
\begin{array}{c}
\chi \\
\vdots \\
\phi \lor \neg \phi \\
\end{array}
\begin{array}{c}
\vdots \\
\neg (\phi \lor \neg \phi) \\
\end{array}
\]

\[
\begin{array}{c}
\phi \lor \neg \phi \\
\end{array}
\begin{array}{c}
\bot \\
\phi \lor \neg \phi \\
\end{array}
\]

[1] $\bot$ \quad $\phi \lor \neg \phi$ \quad \text{RAA}

Now we must derive $\phi \lor \neg \phi$ from $\neg (\phi \lor \neg \phi)$. 
Again, $\neg e$ may be able to help. We need $\phi \lor \neg \phi$ so we assume $\phi$ and then use $\lor i_1$.

\[
\frac{\phi \lor \neg \phi}{\phi \lor \neg \phi} \quad \frac{\neg (\phi \lor \neg \phi) \quad \chi}{\neg e}
\]

Now of course we have a new hypothesis to discharge.
An application of $\neg i$ can be used to discharge hypothesis 2 and yields $\neg \phi$.

Now the goal is to derive $\phi \lor \neg \phi$ from $\neg \phi$. This is easily done with the rule $\lor i_2$. 

$\phi \lor \neg \phi$ 

$[2] \frac{\bot}{\neg \phi} \quad \neg i$

$\vdots$

$\phi \lor \neg \phi$ 

$\frac{\bot}{\phi \lor \neg \phi} \quad \neg e$

$\vdots$

$\frac{\neg (\phi \lor \neg \phi)}{\bot} \quad \neg e$

$[1] \frac{\phi \lor \neg \phi}{\bot} \quad \text{RAA}$
This completes the proof.

\[
\begin{align*}
\lnot \vdash \phi \\
\frac{\phi}{\phi \lor \neg \phi} \lor i_1 \\
\frac{[\neg (\phi \lor \neg \phi)]}{\neg e} \\
\frac{\bot}{\neg \phi} \neg i \\
\frac{\phi \lor \neg \phi}{\phi \lor \neg \phi} \lor i_2 \\
\frac{[\neg (\phi \lor \neg \phi)]}{\neg e} \\
\frac{\bot}{\phi \lor \neg \phi} \text{ RAA}
\end{align*}
\]

This is the shortest natural deduction proof of excluded middle the author knows of.
Now we derive the sequent \( \vdash \phi \lor \neg \phi \) having no assumptions.
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The only rule that applies here is $\lor R$.

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\frac{\vdash \phi, \neg \phi}{\vdash \phi \lor \neg \phi} \quad \lor r
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At this point we have two goals, it is enough to prove either one.

The only rule that applies is \( \neg R \).

\[
\frac{\phi \vdash \phi}{\vdash \phi, \neg \phi} \quad \neg R
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$$\begin{align*}
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\end{align*}$$

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The only rule that applies is $\neg R$.

$$\begin{align*}
\phi \vdash \phi \\
\vdash \phi, \neg \phi \\
\vdash \phi \lor \neg \phi
\end{align*}$$

This is an instance of the axiom rule and the proof is complete.

$$\begin{align*}
\phi \vdash \phi \\
\vdash \phi, \neg \phi \\
\vdash \phi \lor \neg \phi
\end{align*}$$
\[\begin{align*}
\text{\(N\) derivation of GDM} \\
\begin{align*}
\frac{\exists \quad \forall}{\phi \land \psi} & \quad \land i \quad \frac{\chi}{[(\phi \land \psi) \Rightarrow \sigma]} \Rightarrow e \\
\frac{\phi \Rightarrow \sigma}{\phi \Rightarrow \sigma} & \Rightarrow i \\
\frac{\phi \Rightarrow \sigma \lor (\psi \Rightarrow \sigma)}{\neg((\phi \Rightarrow \sigma) \lor (\psi \Rightarrow \sigma))} & \Rightarrow e \\
\frac{\neg e}{\psi \Rightarrow \sigma} & \Rightarrow i \\
\frac{\phi \Rightarrow \sigma \lor (\psi \Rightarrow \sigma)}{\neg((\phi \Rightarrow \sigma) \lor (\psi \Rightarrow \sigma))} & \Rightarrow e \\
\frac{\phi \Rightarrow \sigma \lor (\psi \Rightarrow \sigma)}{\neg e} & \Rightarrow i \\
\frac{\phi \Rightarrow \sigma \lor (\psi \Rightarrow \sigma)}{\neg e} & \Rightarrow i
\end{align*}
\end{align*}\]
Sequent derivation of GDM

\[
\frac{\phi, \psi \vdash \phi, \sigma}{\phi, \psi \vdash \phi \land \psi, \sigma} \quad \text{Ax}
\]

\[
\frac{\phi, \psi \vdash \psi, \sigma}{\phi, \psi \vdash \phi \land \psi, \sigma} \quad \land \text{R}
\]

\[
\frac{\phi \vdash \phi \land \psi, \psi \Rightarrow \sigma}{\phi \vdash \phi \land \psi, \psi \Rightarrow \sigma} \quad \Rightarrow \text{R}
\]

\[
\frac{\sigma, \phi \vdash \sigma, \psi \Rightarrow \sigma}{\sigma, \phi \vdash \sigma, \psi \Rightarrow \sigma} \quad \Rightarrow \text{L}
\]

\[
\frac{(\phi \land \psi) \Rightarrow \sigma, \phi \vdash \sigma, \sigma \Rightarrow \psi}{(\phi \land \psi) \Rightarrow \sigma, \phi \vdash \sigma, \sigma \Rightarrow \psi} \quad \Rightarrow \text{R}
\]

\[
\frac{(\phi \land \psi) \Rightarrow \sigma, \phi \vdash \sigma, \psi \Rightarrow \sigma}{(\phi \land \psi) \Rightarrow \sigma, \phi \vdash \sigma, \psi \Rightarrow \sigma} \quad \Rightarrow \text{R}
\]

\[
\frac{(\phi \land \psi) \Rightarrow \sigma, \phi \vdash \sigma, \psi \Rightarrow \sigma}{(\phi \land \psi) \Rightarrow \sigma, \phi \vdash \sigma, \psi \Rightarrow \sigma} \quad \Rightarrow \text{L}
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\[
\frac{\phi \land \psi \Rightarrow \sigma}{\Rightarrow R}
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\frac{((\phi \land \psi) \Rightarrow \sigma)}{\Rightarrow R}
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Against Sequents

- It is often claimed that natural deduction proofs reflect the mathematical thought process but that sequents do not.
  - I claim it is the rules, not the tree structure of a proof, that serves to “explain” the logical laws.
  - Consider disjunction: “To prove $\phi \lor \psi$ prove $\psi$ or prove $\psi$.”

\[
\begin{align*}
\text{Natural Deduction} & \quad \text{Sequent Calculus} \\
\phi & \quad \frac{\psi}{\phi \lor \psi} \quad \frac{\phi \lor \psi}{\phi \lor \psi} \\
\phi \lor \psi & \quad \phi \lor \psi & \quad \frac{\Gamma \vdash \Delta_1, \phi, \psi, \Delta_2}{\Gamma \vdash \Delta_1, \phi \lor \psi, \Delta_2} \quad \text{(\lor R)}
\end{align*}
\]

- The sequent rule works perfectly well as an explanation.
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\text{Natural Deduction} \quad \begin{array}{c}
\dfrac{\phi}{\phi \lor \psi} \quad \lor i_1 \\
\dfrac{\psi}{\phi \lor \psi} \quad \lor i_2
\end{array}
\quad \text{Sequent Calculus} \quad \dfrac{\Gamma \vdash \Delta_1, \phi, \psi, \Delta_2}{\Gamma, \vdash \Delta_1, \phi \lor \psi, \Delta_2} (\lor R)
\]

- The sequent rule works perfectly well as an explanation.

- Sequent proofs require too much writing.
  - Not in superintuitionistic cases (see examples above).
  - Ink is cheap.
  - With tool support this claim is moot.
  - It is perfectly OK to elide unneeded formulas in the antecedent.

\[
\dfrac{\Gamma' \vdash \Delta}{\Gamma, \vdash \Delta} \quad \text{(Thin)} \quad \text{where } \Gamma' \subseteq \Gamma
\]
Against Sequents

- Multiple formulas on the right are hard to motivate.
  - The sequent semantics and $\lor R$ are justification enough.
    
    \[
    [\Gamma \vdash \Delta] \overset{\text{def}}{=} (\bigwedge_{\phi \in \Gamma} \phi) \Rightarrow \bigvee_{\psi \in \Delta} \psi
    \]

  - Velleman (a best selling book on informal proof methods) uses them.

- Cut is hard to motivate?
  - I find this one hard to understand.

- Cut is not needed though it can be convenient in predicate logic proofs.
  - More conveniently, we add a lemma rule for previously proved theorems.
    
    \[
    \Gamma, \phi \vdash \Delta \quad \Gamma \vdash \Delta \quad (\text{Lemma})
    \]
Against Sequents

- Multiple formulas on the right are hard to motivate.
  - The sequent semantics and $\lor R$ are justification enough.
    \[
    \text{def} = (\bigwedge_{\phi \in \Gamma} \phi) \Rightarrow \bigvee_{\psi \in \Delta} \psi
    \]
  - Velleman (a best selling book on informal proof methods) uses them.

- Cut is hard to motivate?
  - I find this one hard to understand.
    \[
    \Gamma \vdash \phi, \Delta \quad \Gamma, \phi \vdash \Delta \quad \text{(Cut)}
    \]
  - Cut is not needed though it can be convenient in predicate logic proofs.
  - More conveniently, we add a lemma rule for previously proved theorems.
    \[
    \Gamma, \phi \vdash \Delta \quad \text{(Lemma) where} \vdash \phi
    \]
For Sequents

- The provide a decision procedure for propositional logic.
- Counter-examples are easily generated from failed proofs.
- Students gain confidence as they become more adept at manipulating the formalism.
- There is no question “Is this a proof?”
- Curry-Howard still holds (for $\mathcal{LJ}$ proofs.)
- It is easy to identify the superintuitionistic theorems – which ones necessarily have two formulas on the right at some point in the derivation.
Propositional Logic is Easy!

Teaching propositional logic using Natural Deduction is a bit like teaching arithmetic using Roman numerals.

You could force students to suffer through it, but aren’t Arabic numerals better suited to the task?