Formalizing Simplicial Topology in ACL2

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AGENDA

- Introduction
- Simplicial Topology in ACL2
- A developed example
  - A direct proof
  - A proof based on abstract reduction systems
- Conclusions and further work

Introduction

The Common Lisp system
Kenzo
to compute in Algebraic Topology

- tested but … not always

- mechanized proofs in Isabelle for some theoretical algorithms used in **Kenzo**

- distance from the Kenzo code to the theories and proofs in Isabelle

Introduction

Our idea: using ACL2 to verify the actual Kenzo programs

But ... Kenzo uses higher order functional programming

How could we increase the reliability of Kenzo ?

Our proposal:

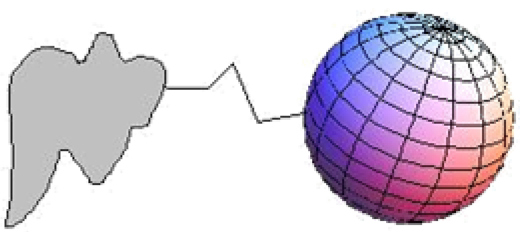
Choose, reprogram and verify in ACL2 first-order fragments of Kenzo related with Simplicial Topology

Simplicial Topology in ACL2

Abstract topological spaces replaced by simplicial sets (combinatorial artifacts)

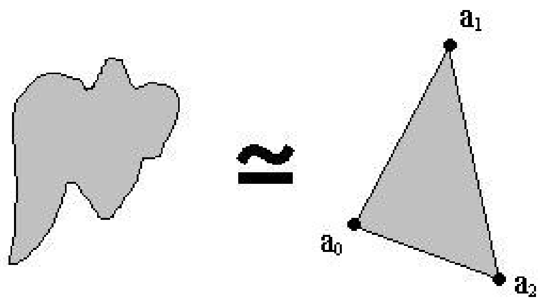
- Motivation: algebraic invariants are computed in an easier way

<u>Example</u>: topological space



Simplicial Topology in ACL2

Triangulating the space



Triangle can be described by (a_0, a_1, a_2) where the faces are obtained in this way:

$$\partial_0(a_0, a_1, a_2) = (a_1, a_2)$$
$$\partial_1(a_0, a_1, a_2) = (a_0, a_2)$$
$$\partial_2(a_0, a_1, a_2) = (a_0, a_1)$$

$$\boxed{\partial_i \, \partial_j = \partial_{j-1} \, \partial_i \quad \text{if } i<j}$$

The faces of each edge are defined analogously :

$$\partial_0(a_1, a_2) = (a_2)$$
$$\partial_1(a_1, a_2) = (a_1)$$

.

.

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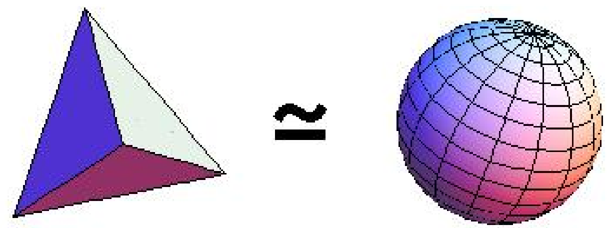
Simplicial Topology in ACL2

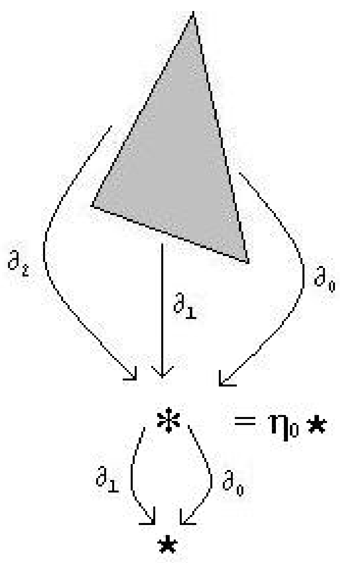
4 vertices
6 edges
4 triangles

14 elements



Triangle faces:

$\partial_0 x = \partial_1 x = \partial_2 x = \eta_0(*)$.



1 triangle

1 collapsing point

2 elements

$\eta_0(*)$ is called **degeneration** of $\star$

Simplicial Topology in ACL2

$\eta_0 (a_0, a_{1,}, a_2) := (a_0 , a_0 , a_1 , a_2)$

$\eta_1 (a_0, a_{1,}, a_2) := (a_0 , a_1 , a_1 , a_2)$

$\eta_2 (a_0, a_{1,}, a_2) := (a_0 , a_1 , a_2 , a_2)$

The operator η_i is repeating the i-th element in the list

Simplicial Topology in ACL2

Definition. A ***simplicial set*** K consists of a graded set $\{K_q\}_{q \in N}$ and, for each pair of integers (i,q) with 0<=i<=q, ***face*** and ***degeneracy*** maps, $\partial_i : K_q \rightarrow K_{q-1}$ and $\eta_i : K_q \rightarrow K_{q+1}$, satisfying the simplicial identities:

$$
\begin{array}{ll}
\partial_i \partial_j = \partial_{j-1} \partial_i & \text{if } i<j \\
\eta_i \eta_j = \eta_{j+1} \eta_i & \text{if } i<=j \\
\partial_i \eta_j = \eta_{j-1} \partial_i & \text{if } i<j \\
\partial_i \eta_j = Id & \text{if } i=j \text{ or } i=j+1 \\
\partial_i \eta_j = \eta_j \partial_{i-1} & \text{if } i>j+1
\end{array}
$$

The elements of K_q are called ***q-simplices***

A q-simplex x is **degenerate** if $x = \eta_i y$ with $y \in K_{q-1}$, $0<=i<q$

Otherwise x is called **non-degenerate**

0-simplices as vertices
Non-degenerate 1-simplices as edges
Non-degenerate 2-simplices as (filled) triangles
Non-degenerate 3-simplices as (filled) tetrahedra
...

Simplicial Topology in ACL2

We focus our studies on the **universal simplicial set Δ**

- *Reason*: Any theorem proved on Δ by using only the equalities of the previous definition will be also true for any other simplicial set K

In ACL2

- a q-simplex of Δ is any ACL2 list of length q
- face operators are defined by means of the function `(del-nth i l)` which eliminates the i-th element in the list l
- degeneracy operators are defined by means of the function `(deg i l)` which repeats the i-th element in the list l

We consider the simplicial set freely generated from the set of all ACL2 objects

An example

Theorem 1. Let K be a simplicial set. Any degenerate n-simplex $x \in K_n$ can be expressed in a $\boxed{\text{unique}}$ way as a (possibly) iterated degeneracy of a non-degenerate simplex y in the following way:

$$x = \eta_{jk} \dots \eta_{j1} y$$

with $y \in K_r$, $k = n-r > 0$, $\boxed{0 <= j_1 < ... <= j_k < n}$

Thinking in ACL2

- A non-degenerate simplex in Δ is a list where any two consecutive elements are different
- A simplex in Δ can be represented as a pair of lists, the first one a list of natural numbers (degeneracy list) and the second one any ACL2 list.

Theorem 2. Any ACL2 list l can be expressed in a unique way as a pair (dl,l') such that l= degenerate (dl,l') with l' without two consecutive elements equal and dl a strictly increasing degeneracy list.

A direct ACL2 proof of theorem 2

```
(defun generate (l)
  (if (or (endp l) (endp (cdr l)))
              (cons nil l)
    (let ((gencdr (generate (cdr l))))
      (if (equal (first l) (second l))
          (cons (cons 0 (add-one (car gencdr)))
                (cdr gencdr))
        (cons (add-one (car gencdr))
              (cons (car l) (cdr gencdr)))))))

(defthm existence
  (let ((gen (generate l)))
    (and (canonical gen)
         (equal (degenerate (car gen) (cdr gen)) l))))
```

A direct ACL2 proof of theorem 2

```
(defthm uniqueness-main-lemma
  (implies (canonical (cons l1 l2))
           (equal (generate (degenerate l1 l2))
                  (cons l1 l2))))
```

The lists obtained after rewriting `(generate (degenerate l1 l2))` in `(generate (degenerate (cdr l1) (deg (car l1) l2)))` do not satisfy the hypotheses of the theorem. Not possible to apply a simplified induction scheme.

```
(defthm uniqueness
  (implies
    (and (canonical p1) (canonical p2)
         (equal (degenerate (car p1) (cdr p1)) l)
         (equal (degenerate (car p2) (cdr p2)) l))
    (equal p1 p2)))
```

An abstract reduction systems approach

An alternative proof because:

- The direct proof does not explicitly use the face operators
- The direct proof is not directly based on the combinatorial properties which relate the face and degeneracy maps

Idea:

To consider the elimination of a consecutive repetition in a list (face operator) as a simple **reduction step**

Another type of **reduction step** to “fix” disorders in the degeneracy list

An abstract reduction systems approach

Formalizing:

- We define the reduction system $\rightarrow_S$ where:
  - the set of *S*-terms is the set of pairs (l_1, l_2) where

    l_1 a list of natural numbers

    l_2 any list
  - two types of rules are considered in $\rightarrow_S$:

• ***o-reduction****:* if the list l_1 has a "disorder" at position i, i.e., $l_1(i) >= l_1(i+1)$, then $(l_1, l_2) \rightarrow_S (l'_1, l_2)$, where $l'_1(i) = l_1(i+1)$ and $l'_1(i+1) = l_1(i)+1$, (here l(j) denotes the j-th element of l)

$$\eta_i \, \eta_j = \eta_{j+1} \, \eta_i \qquad \text{if } i<=j$$

•***r-reduction****:* if at index i there is a repetition in l_2 , i.e. , $l_2(i) = l_2(i+1)$, then $(l_1, l_2) \rightarrow_S (l'_1, l'_2)$, where $l'_1 = \text{cons}(i, l_1)$ and $l'_2 = \text{del-nth}(i, l_2)$

$$\partial_i \, \eta_j = \text{Id} \qquad \text{if } i=j \text{ or } i=j+1$$

An abstract reduction systems approach

- Modeling our reduction system in ACL2

- Model $\rightarrow_S$ in the framework of **Ruiz Reina's** ACL2 formalization about abstract reduction systems

Operators are pairs (t,i) where
t is 'o or 'r
i is the position in the list where the corresponding reduction takes place

The **relation** $\rightarrow_S$ is represented by two functions :

```
(s-legal x op)
(s-reduce-one-step x op)
```

They suffice to represent a reduction and other related concepts:
noetherianity, equivalence closures, normal forms or confluence

An abstract reduction systems approach

- We proved that the reduction is noetherian *(there is no infinite sequence of S-reductions)* using a suitable lexicographic measure

- We defined a function to compute a normal form with respect to $\rightarrow_S$

```
(defun s-normal-form (x)
  (let ((red (s-reducible x)))
    (if red
          (s-normal-form (s-reduce-one-step x red))
      x)))
```

- We proved that $\rightarrow_S$ is locally confluent *(whenever there is a local peak, there is a valley)*

```
(defthm local-confluence
  (implies (and (s-equiv-p x y p) (local-peak-p p))
     (and (s-equiv-p x y (s-transform-local-peak p))
         (steps-valley (s-transform-local-peak p)))))
```

- Newman's Lemma: every noetherian and locally confluent reduction is convergent. It means that two equivalent elements have a common normal form

```
(defthm s-reduction-convergent
  (implies (s-equiv-p x y p)
     (equal (s-normal-form x) (s-normal-form y))))
```

An abstract reduction systems approach

- The main relation between $\to_S$ and the function degenerate is given by

a) If $(l_1, l_2) \to_S (l_3, l_4)$, then degenerate (l_1, l_2)= degenerate (l_3, l_4)

b) If degenerate (l_1, l_2)=l then (nil,l)$=_S (l_1, l_2)$

```
(defthm degenerate-s-equivalent
   (implies …
      (s-equiv-p (cons l m)
                 (cons nil (degenerate l m))
                 (degenerate-steps l m))))
```

- We define (generate l) as (s-normal-form (cons nil l)))

- We prove the theorems existence and uniqueness exactly as stated previously

- Corollary: both definitions of generate are equivalent

Conclusions

- We have presented some ideas to apply ACL2 in Simplicial Topology. Main contributions:
  - ✓analysis of feasibility
  - ✓relation of ACL2 proofs in Simplicial Topology with abstract rewriting systems
- Increase the reliability of a real Computer Algebra program (Kenzo)

Further work

- Formalize and prove more difficult results from Simplicial Topology in ACL2
- ACL2 proof of the Eilenberg-Zilber theorem