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Munich, Germany

Formal Verification of Optical Quantum Flip Gate

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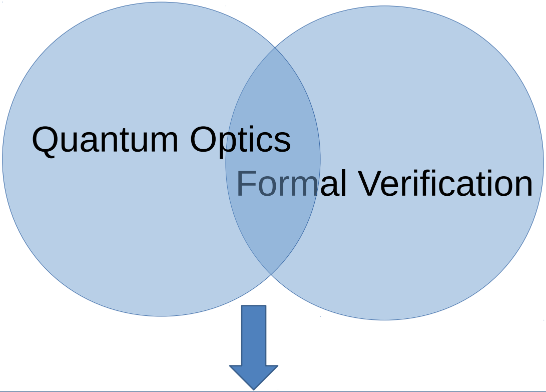
Outline

Motivation



Optical Quantum Computers Verification

Context

Global context:

- Research program carried out at Concordia University
- Objective: formal verification of *optical systems* (e.g., fiber optic, optic circuits, quantum computers, etc.)

To do so, formalization of various theories of optics:

- Ray optics
- Electromagnetic optics
- Quantum optics

Outline

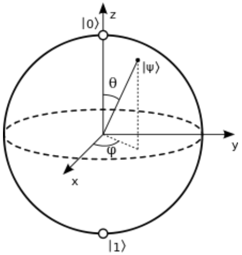
Quantum Computers

Classical Bit

Classical bit = 0 or 1

Qubit

Quantum bit = "mix" of $|0\rangle$ and $|1\rangle \rightarrow \delta|0\rangle + \beta|1\rangle$



Implementation:

- Physical implementations of Qubits: photons, electrons or ions
- *Photon*-based implementations are the most promising

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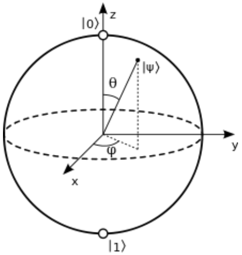

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- Physical implementations of Qubits: photons, electrons or ions
- *Photon*-based implementations are the most promising

Photon-based implementation of Qubits

Coherent light is used to represent quantum bits:

Coherent Light Qubit

- Qubit = coherent light with states $|0\rangle$ and $|\alpha\rangle$
- $|0\rangle$ and $|\alpha\rangle$ represent $|0\rangle$ and $|1\rangle$, respectively

Photon-based implementation of Qubits

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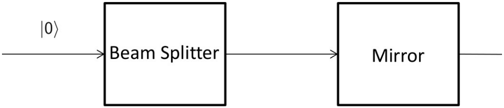
Coherent Light Quantum Flip Gate

- Quantum flip gate, converts $|0\rangle$ into $|1\rangle$ and vice versa
- Implemented as a *beam splitter* and a *phase conjugating mirror*.



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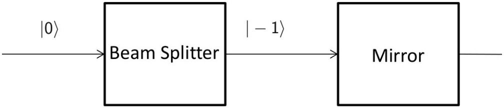

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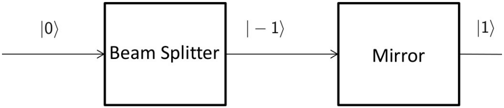

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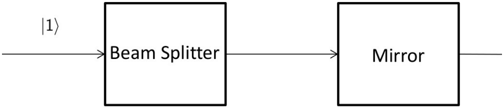

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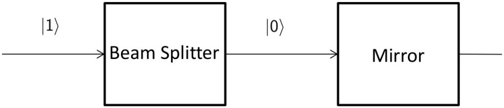

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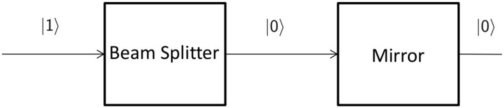
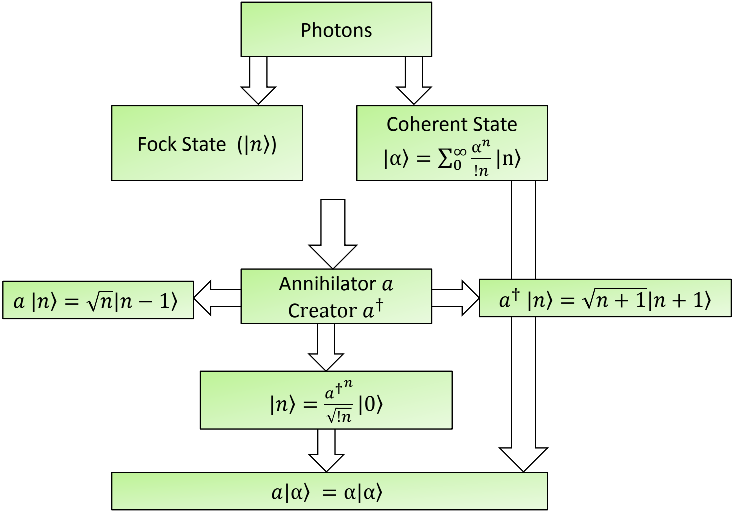


Quantum Optics: Overview



Outline

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Mathematics Prerequisites

- Complex functions spaces (`cfun`). NFM13
- Infinite summation over `cfun`. NFM14
- Infinite summation over quantum operators.
- Exponentiation of quantum operators.

Infinite Summation over Operators: Definition

$\sum_{n=0}^{\infty} f_n = g_n \Leftrightarrow \forall x \sum_{n=0}^{\infty} f_n(x) = g_n(x)$

Definition

- *Specification:*
  cop_sums (s, inprod) f l (from 0) $\Leftrightarrow$ $\forall$x. x IN s $\Rightarrow$
  cfun_sums (s, inprod) (λn.(f n) x) (l x) (from 0)
- *Hilbert operator to make a function out of it:*
  cop_infsum innerspc s f = @l. cop_sums innerspc f l s
- *Existence predicate:*
  cop_summable innerspc s f = $\exists$l. cop_sums innerspc f l s

Infinite Summation: Properties

" $\sum_{n=0}^{\infty}(f_n + g_n) = \sum_{n=0}^{\infty} f_n + \sum_{n=0}^{\infty} g_n$"

Theorem (Linearity of infinite summation - 1)

$\forall$ f g innerspc.
cop_summable innerspc s f $\wedge$ cop_summable innerspc s g $\Rightarrow$
cop_infsum innerspc s (λn. fn + gn) =
cop_infsum innerspc s f + cop_infsum innerspc s g

" $\sum_{n=0}^{\infty}(a.f_n) = a.\sum_{n=0}^{\infty} f_n$"

Theorem (Linearity of infinite summation - 2)

$\forall$ f innerspc a. cop_summable innerspc s f $\Rightarrow$
cop_infsum innerspc s (λn. a % f n)
= a % cop_infsum innerspc s f

% = multiplication by a scalar

Commutativity of Fun. Inf. Summation with Linear Operators

Definition (Linearity)

is_linear_cop s (op : cop) $\Leftrightarrow$
$\forall$x y.x IN s $\wedge$ y IN s $\Rightarrow$ op (x + y) = op x + op y
$\wedge$ $\forall$a. op (a % x) = a % (op x)

"if *op* linear & bounded: $\sum_{n=0}^{\infty}(op\ (f_n)) = op\ (\sum_{n=0}^{\infty} f_n)$"

Theorem (Commutativity of Inf. Summation with Linear Op.)

$\forall$f h s innerspc.
is_linear_cop s h $\wedge$ is_bounded innerspc h
$\Rightarrow$ cfun_infsum innerspc s (λn. h(f n))
= h (cfun_infsum innerspc s f)

Exponentiation of Quantum Operators

"$e^{op} = \sum_{i=0}^{\infty} \frac{op^n}{!n}$"

Definition

$\texttt{cop\_exp innerspc (op : cfun} \rightarrow \texttt{cfun)} \Leftrightarrow$
$\quad \texttt{cop\_infsum innerspc (from 0) (}\lambda\texttt{n. } \frac{1}{\texttt{!n}} \texttt{ \% (op pow n)}$

" $e^{constantly\ \ null\ \ operator} = identity$"

Theorem

$\forall \texttt{s inprod x. x IN s} \wedge \texttt{is\_inner\_space (s, inprod)} \Rightarrow$
$\quad \texttt{cop\_exp (s, inprod) cop\_zero x = x}$

" $e_{op}^{a.op}(x) = e_{\mathbb{C}}^{a}.op(x)$"

Theorem

$\forall \texttt{ s inprod a x. x IN s} \wedge \texttt{is\_inner\_space (s, inprod)} \Rightarrow$
$\quad \texttt{(cop\_exp (s, inprod) (}\lambda\texttt{y. a\%y)) x = cpow a \% x}$

Outline

Coherent Light: Definition

Coherent light $\Rightarrow$ number of photons follows Poisson distribution at any time

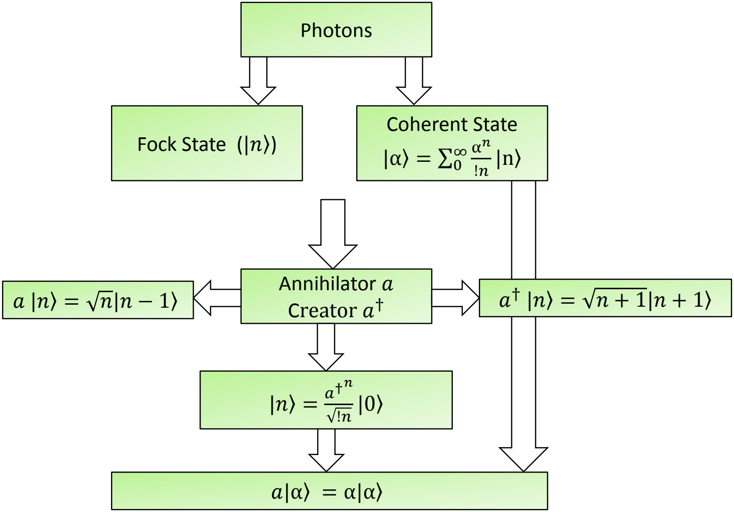
More precisely: state of a coherent light $= |\alpha\rangle$ where $|\alpha|^2$ is the distribution parameter, i.e., the number of expected photons.

Definition

$$\texttt{coherent sm } \alpha = \exp(-\frac{|\alpha|^2)}{2}))\% \; \textcolor{red}{\texttt{cfun\_infsum}} \; (\texttt{s}, \texttt{inprod}) \; (\texttt{from } 0) \; (\lambda \texttt{n}. \frac{\alpha^{\texttt{n}}}{\sqrt{\texttt{n!}}} \% (\textcolor{red}{\texttt{fock}} \; \texttt{sm n}))$$

fock sm n = state where we have *n* photons
$\rightarrow$ defined using the creation operator and the vaccum state (...)
$\rightarrow$ themselves defined using the definition of sm (...)

Quantum Optics: Overview (Recall)



Coherent Light: Property

Theorem (Expression using the displacement operator)

$$(\forall \mathtt{n}.\mathtt{creat\_of\_sm\ sm\ (fock\ sm\ n)} \neq \mathtt{cfun\_zero}))$$
$$\wedge\ \mathtt{cfun\_summable\ (s, inprod)\ (from\ 0)}(\lambda \mathtt{n}.\frac{\alpha\ \mathtt{pow\ n}}{\sqrt{\mathtt{!n}}}\ \mathtt{\%\ fock\ sm\ n})$$
$$\mathtt{is\_sm\ sm} \wedge\ \mathtt{exp\_summable\ (qspc\_of\_sm\ sm)}\ (\alpha\ \mathtt{creat\_of\_sm\ sm})$$
$$\Rightarrow \mathtt{coherent\ sm}\ \alpha = (\mathtt{disp\ sm}\ \alpha)\ \mathtt{vac}$$

vac = lowest energy coherent state ("vaccum")

Displacement Operator

$$D(\alpha) = e^{\alpha \hat{a}^\dagger} \; e^{-\alpha^* \hat{a}} \; e^{[\alpha \hat{a}^\dagger, \alpha^* \hat{a}]}$$

$\hat{a}$ = creation operator (adds a level of energy/photon to a quantum system)
$[a, b]$ = commutator between a and b, i.e., $a \circ b - b \circ a$

Definition (Displacement Operator)

disp sm α =
  (cop_exp sm (α % creat_of_sm sm) $**$
    cop_exp sm ($-$(cnj α) % a_of_sm sm) $**$
     cop_exp sm ((α % creat_of_sm sm) com ((cnj α) % a_of_sm sm)

Main interest of the displacement operator:
easily implemented (physically) using a beam splitter

Outline

Optical Flip Gate (Recall)

Coherent Light Qubit

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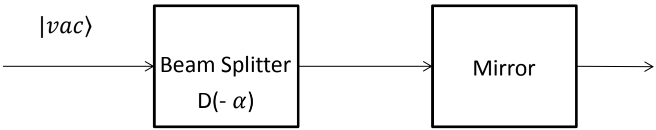

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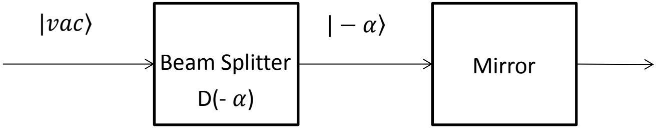

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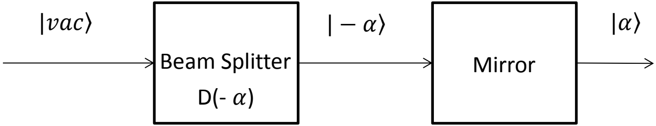

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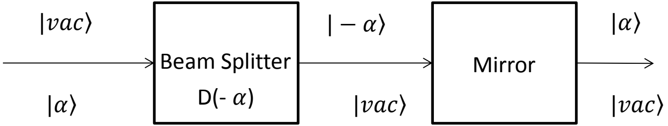

Beam Splitter over Coherent states

Theorem (Beam splitter over $|1\rangle$)

$$\wedge\ (\forall \mathtt{x}\ \mathtt{op}.\ \mathtt{is\_linear\_cop}\ \mathtt{op} \wedge \mathtt{x}\ \mathtt{IN}\ \mathtt{s} \Rightarrow$$
$$(\mathtt{cop\_exp}\ (\mathtt{s}, \mathtt{inprod})\ (-\mathtt{op}) **\ \mathtt{cop\_exp}\ (\mathtt{s}, \mathtt{inprod})\ (\mathtt{op}))\ \mathtt{x} = \mathtt{x}$$
$$\Rightarrow \mathtt{disp}\ \mathtt{sm}\ (-\alpha)\ (\mathtt{coherent}\ \mathtt{sm}\ \alpha) = \mathtt{vac}$$

Theorem (Beam splitter over $|0\rangle$)

$$\wedge\ (\forall \mathtt{x}\ \mathtt{op}.\ \mathtt{is\_linear\_cop}\ \mathtt{op} \wedge \mathtt{x}\ \mathtt{IN}\ \mathtt{s} \Rightarrow$$
$$(\mathtt{cop\_exp}\ (\mathtt{s}, \mathtt{inprod})\ (-\mathtt{op}) **\ \mathtt{cop\_exp}\ (\mathtt{s}, \mathtt{inprod})\ (\mathtt{op}))\ \mathtt{x} = \mathtt{x}$$
$$\Rightarrow \mathtt{disp}\ \mathtt{sm}\ (-\alpha)\ (\mathtt{coherent}\ \mathtt{sm}\ \mathtt{vac}) = -\alpha$$

Note: These proofs requires Baker-Campbell-Hausdorf theorem
→ assumed in this work

Mirror over Coherent states

Definition (Mirror)

$$\texttt{mirror sm} = \texttt{cop\_exp (s, inprod) (i}\pi\texttt{ \% n\_of\_sm sm)}$$

Theorem (Main mirror property)

$$\texttt{mirror\_summable sm} \wedge \texttt{is\_bounded (qspc\_of\_sm sm) (mirror sm)} \wedge (\forall \texttt{n.creat\_of\_sm sm (fock sm n)} \neq \texttt{cfun\_zero)}) \Rightarrow \texttt{mirror sm (coherent sm } \alpha\texttt{)} = \texttt{coherent sm } (-\alpha)$$

Theorem (Main mirror property)

$$\texttt{mirror\_summable sm} \wedge \texttt{is\_bounded (qspc\_of\_sm sm) (mirror sm)} \wedge (\forall \texttt{n.creat\_of\_sm sm (fock sm n)} \neq \texttt{cfun\_zero)}) \Rightarrow \texttt{mirror sm (coherent sm vac)} = \texttt{coherent sm vac}$$

Formal Flip Gate Verification

Definition (Flip Gate)

$$\texttt{flip\_gate}\ \alpha\ \texttt{sm} = (\texttt{mirror sm})\ **\ (\texttt{disp sm}\ (-\alpha))$$

Main result of this work:
"flip gate applied to $|1\rangle$ returns $|0\rangle$"
and
"flip gate applied to $|0\rangle$ returns $|1\rangle$"

Theorem

$$\begin{aligned}
&(\texttt{coherent sm}\ \alpha \neq \texttt{cfun_zero})\wedge\\
&\wedge\ (\texttt{cop_exp}\ (\texttt{s}, \texttt{inprod})\ (-\texttt{op})\ **\ \texttt{cop_exp}\ (\texttt{s}, \texttt{inprod})\ (\texttt{op}))\ \texttt{x} = \texttt{x})\\
&\wedge\ \texttt{mirror_summable sm} \wedge \texttt{is_bounded}\ (\texttt{qspc_of_sm sm})\ (\texttt{mirror sm})\\
&\quad \Rightarrow\ (\texttt{flip_gate}\ \alpha\ \texttt{sm})\ (\texttt{coherent sm}\ \alpha) = \texttt{vac}\\
&\quad\ \wedge\ (\texttt{flip_gate}\ \alpha\ \texttt{sm})\ \texttt{vac} = \texttt{coherent sm}\ \alpha
\end{aligned}$$

Outline

Conclusion

- Used HOL Light to formally verify that a quantum-optic-based physical system implements a flip gate *(under reasonable assumptions)*
- Required the formal development of several theories
- Most important fact: we went from the maths foundations to a close-to-practice implementation

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Faculty of Engineering and Computer Science

Thanks!
Questions?

http://hvg.ece.concordia.ca