Towards a Formally Verified Proof Assistant

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July 11, 2014

Our initial motivation

How do we know that our systems are sound? How do we safely extend them?

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Formalization of air traffic controllers

Formal verification of banking protocols

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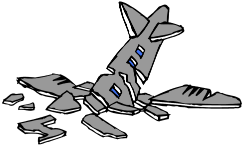
Formalization of air traffic controllers



Formal verification of banking protocols

Our initial motivation

How do we know that our systems are sound? How do we safely extend them?

- Proofs mostly carried out on paper.
- Not carried out in full detail.
- Spread over several papers/PhD theses.
- Precise metatheory, precise account of Nuprl.
- No better way than using a proof assistant.

Our initial motivation

Agda & Coq

➲ 2013/2014: bug in their termination checker.

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Nuprl

Inconsistencies related to types and rules, e.g.,

- Mendler's recursive type,
- LEM is inconsistent with Base

How can we be sure that these rules are valid?

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How can we be sure that these rules are valid?

Nuprl's PER semantics in Coq (and Agda).

Mechanization and Experimentation!

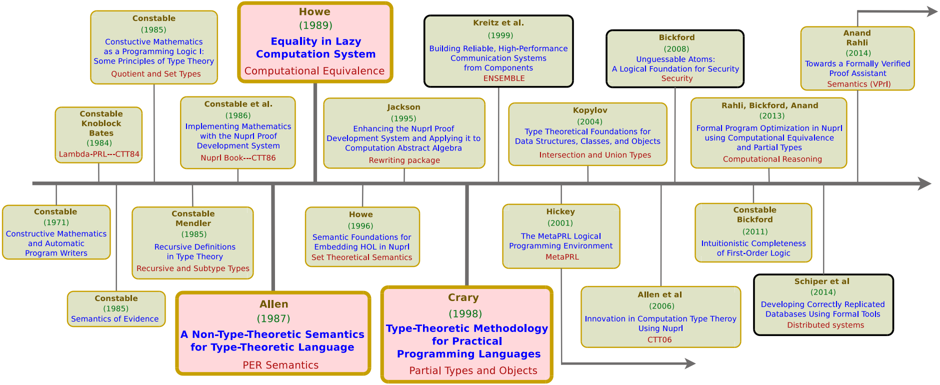
Mechanization

- ➲ Less error prone
- ➲ Easier to propagate changes
- ➲ Positive feedback loop
- ➲ Additive

Experimentation

- ➲ Adding new computations
- ➲ Adding new types
- ➲ Exploring type theory
- ➲ Changing the theory

What do we cover?



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(1971)
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Partial Types and Objects

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Formal Program Optimization in Nuprl using Computational Equivalence and Partial Types
Computational Reasoning

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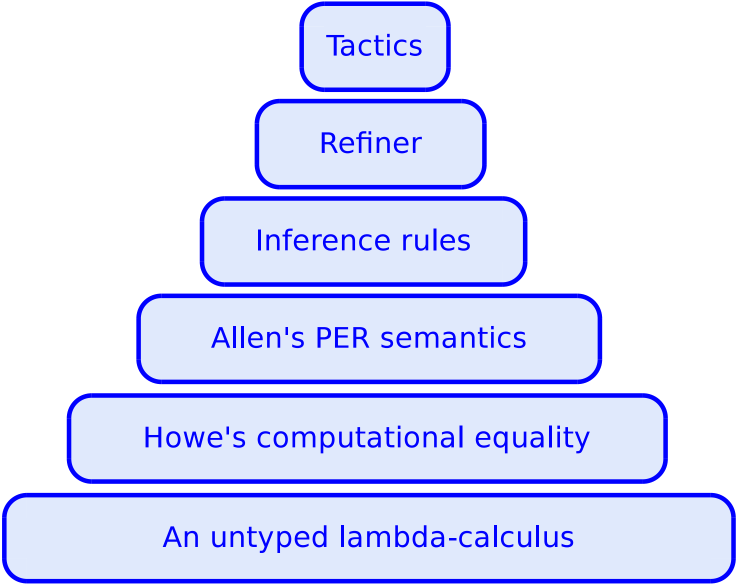
(2014)
Developing Correctly Replicated Databases Using Formal Tools
Distributed systems

Anand Rahli
(2014)
Towards a Formally Verified Proof Assistant
Semantics (VPrl)

Stuart Allen had his own meta-theory that was meant to be meaningful on its own and needs not be framed into type theory. We chose to use Coq and Agda.

Nuprl — Stack



Tactics

Refiner

Inference rules

Allen's PER semantics

Howe's computational equality

An untyped lambda-calculus

Nuprl — Environment

Distributed

Structure editor

Tactic language: Classic ML

Runs in the cloud

Shared library

Nuprl — Types

Equality: $a = b \in T$

Dependent function: $a{:}A \rightarrow B[a]$

Dependent product: $a{:}A \times B[a]$

Universe: $\mathbb{U}_i$

Nuprl — Types

Partial: $\overline{A}$

Intersection: $\cap a{:}A.B[a]$

Subset: $\{a : A \mid B[a]\}$

Computational equivalence: $t_1 \sim t_2$

Image: $\texttt{Img}(A, f)$

PER: $\texttt{per}(R)$, with R a partial equivalence relation.

Nuprl — Types

- **Rich type language facilitates specification**
- **Makes type checking harder**

Inductive types?

- **Using W types.**

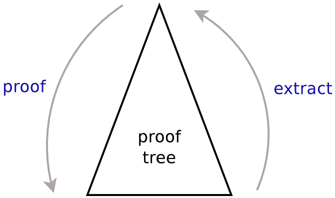
- In Nuprl, we used to define inductive types using Mendler's recursive types. PER semantics?
- We now use Brouwer's bar induction rule to define W types. Validity?

Nuprl — Trusted core

Nuprl's proof engine is called a refiner.

A generic goal directed reasoner:

- **a rule interpreter**
- **a proof manager**



Nuprl — Trusted core

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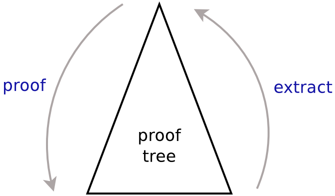
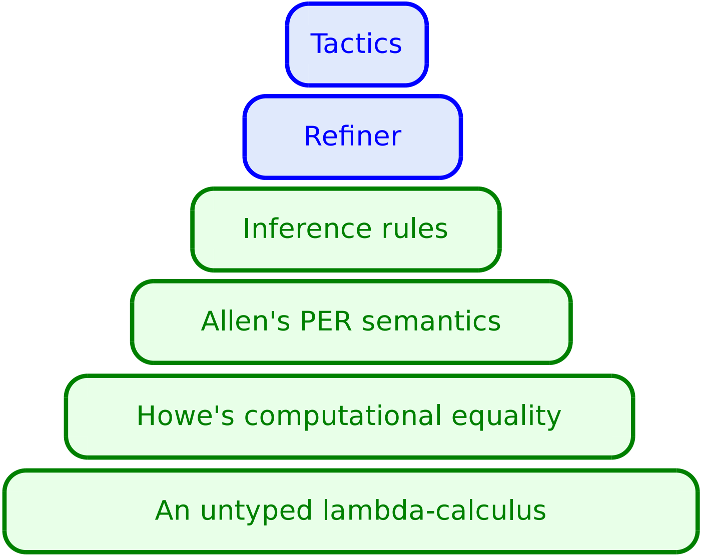


Parameterized by a collection of rules

- **We proved that Nuprl's rules are valid**

- **Next step is to build a verified refiner**

What we implemented in Coq



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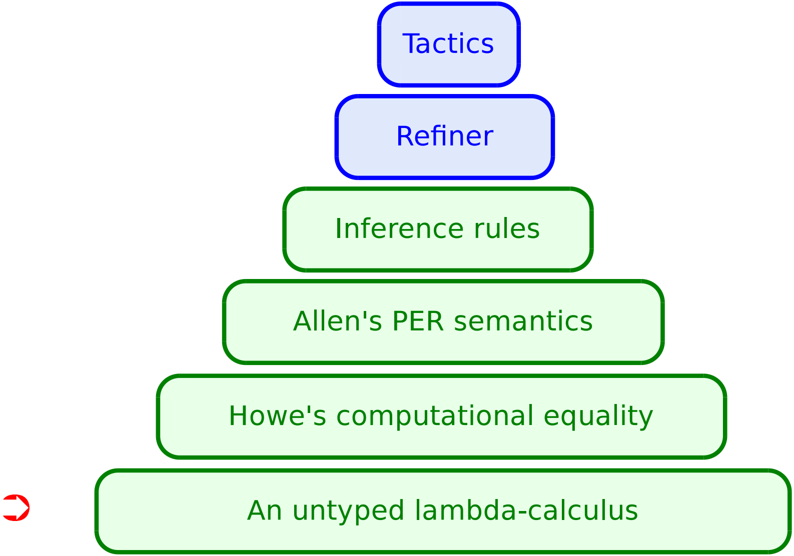

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➲ An untyped lambda-calculus

An untyped lambda-calculus

A "nominal" approach:

```
Inductive NTerm : Set :=
| vterm: NVar → NTerm
| oterm: Opid → list BTerm → NTerm
with BTerm : Set :=
| bterm: list NVar → NTerm → BTerm.
```

For example:

oterm (Can NLambda) [bterm [nvar 0] (vterm (nvar 0))]

represents a λ-term of the form $\lambda x.x$.

An untyped lambda-calculus

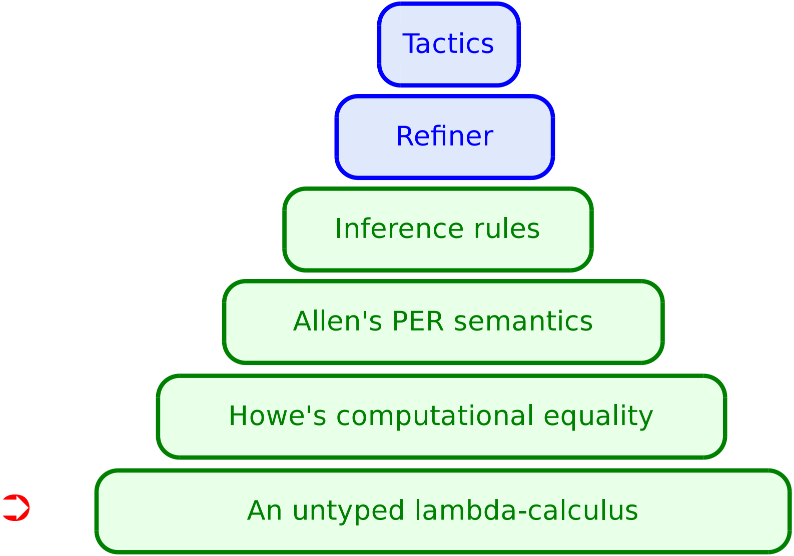
We have the usual computation rules

with a β-reduction rule, pair and injection destructors, a call-by-value operator, a fix operator, exceptions, . . .

Provides a generic framework for defining and reasoning about programming languages using a “nominal” style

➲ **See Abhishek’s LFMTP talk on Thursday**

What we have to implement



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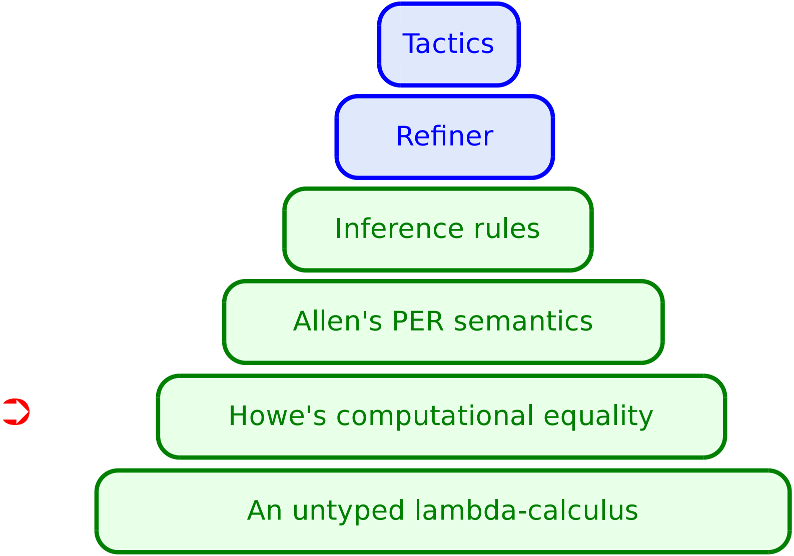

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Howe's computational equality

One can think of approx as the greatest fixpoint of the following operator on binary relations:

```
Definition close_compute
              (R : NTerm→ NTerm →Type)
              (a b : NTerm) : Type :=
programs [ a, b ]
× ∀ (c : CanonicalOp) (as : list BTerm),
  a ⇓ oterm (Can c) as
  → {bs : list BTerm
      & (b ⇓ oterm (Can c) bs)
      × lblift (olift R) as bs }.
```

Howe's computational equality

One would like to define

```
CoInductive approx (a b : NTerm) : Type :=
| approx_fold: close_compute approx a b → approx a b.
```

Unfortunately, because of `cofix`'s conservative productivity checking, we had to use parametrized coinduction.

```
Definition cequiv a b := approx a b × approx b a.
```

approx ($\preceq$) and cequiv ($\sim$) are congruences

Constructive domain theory (Crary)

Let $\perp$ be $\mathtt{fix}(\lambda x.x)$.

Constructive domain theory (Crary)

Let $\bot$ be $\texttt{fix}(\lambda x.x)$.

Least element

$\forall t.$ approx $\bot$ $t.$

Constructive domain theory (Crary)

Let $\perp$ be $\texttt{fix}(\lambda x.x)$.

Least element

$\forall t.$ approx $\perp$ t.

Least upper bound principle

$\forall G\ f.\ G(\texttt{fix}(f))$ is the lub of the (approx) chain $G(f^n(\perp))$ for $n \in \mathbb{N}$.

Constructive domain theory (Crary)

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Least upper bound principle

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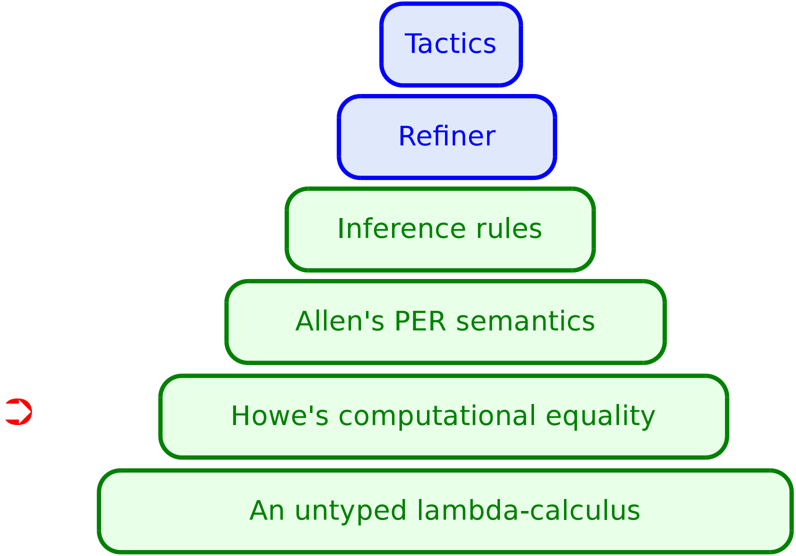
Compactness

if $G(\texttt{fix}(f))$ converges, then there exists a natural number n such that $G(f^n(\perp))$ converges.

What we have to implement



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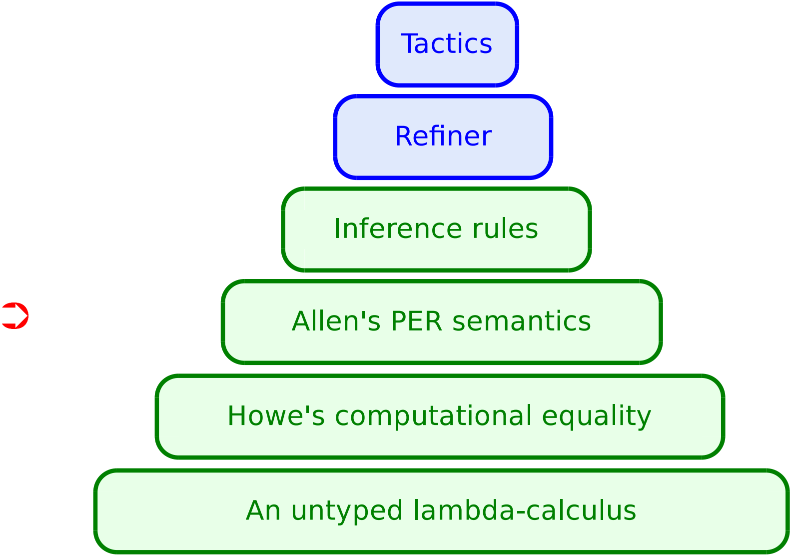

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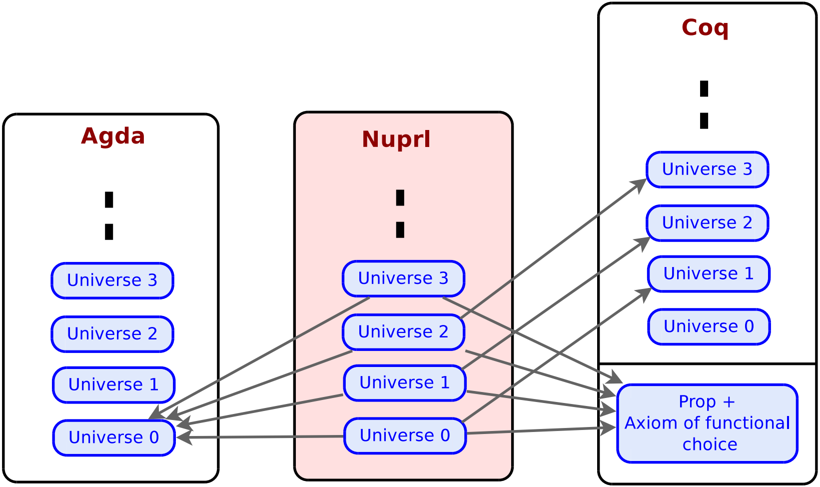

Allen's PER semantics

$$f_1 \equiv f_2 \in x{:}A \to B$$

$$(x{:}A \to B)\ \texttt{type} \wedge \forall a_1, a_2.\ a_1 \equiv a_2 \in A \Rightarrow f_1(a_1) \equiv f_2(a_2) \in B[x \backslash a_1]$$

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$$t_1 \equiv t_2 \in \mathtt{Base}$$

$$t_1 \sim t_2$$

$$\mathtt{Ax} \equiv \mathtt{Ax} \in (a = b \in A)$$

$$(a = b \in A)\ \mathtt{type} \wedge a \equiv b \in A$$

$$t_1 \equiv t_2 \in \overline{A}$$

$$(\overline{A})\ \mathtt{type} \wedge (t_1 \Downarrow \iff t_2 \Downarrow) \wedge (t_1 \Downarrow \Rightarrow t_1 \equiv t_2 \in A)$$

Allen's PER semantics

$$x_1{:}A_1 \rightarrow B_1 \equiv x_2{:}A_2 \rightarrow B_2$$

$$A_1 \equiv A_2 \wedge \forall a_1, a_2.\ a_1 \equiv a_2 \in A_1 \Rightarrow B_1[x_1 \backslash a_1] \equiv B_2[x_2 \backslash a_2]$$

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$$\mathtt{Base} \equiv \mathtt{Base}$$

$$(a_1 = a_2 \in A) \equiv (b_1 = b_2 \in B)$$

$$A \equiv B \wedge (a_1 \equiv b_1 \in A \vee a_1 \sim b_1) \wedge (a_2 \equiv b_2 \in A \vee a_2 \sim b_2)$$

$$\overline{A} \equiv \overline{B}$$

$$A \equiv B \wedge (\forall a.\ a \in A \Rightarrow a\Downarrow)$$

Allen's PER semantics

This definition can be made formal using induction-recursion

Simple induction mechanisms such as in Coq are not enough

➲ Definition is non-strictly-positive

Allen suggests that the definition should be valid because it is "half-positive" (achieved by induction-recursion)

Instead of using induction-recursion, Allen defines ternary relations between types and equalities

➲ Translation of a mutually inductive-recursive definition to a single inductive definition (Capretta).

Allen's PER semantics

Ternary relations

candidate type systems:

$$\mathtt{cts} = \mathrm{CTerm} \to \mathrm{CTerm} \to \mathrm{per} \to \mathtt{Univ}$$

where $\mathrm{per} = \mathrm{CTerm} \to \mathrm{CTerm} \to \mathtt{Univ}$

Allen's PER semantics

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Type constructors

$\mathtt{Definition}\ \mathrm{per\_function}\ (\mathit{ts} : \mathtt{cts}) : \mathtt{cts} := \ldots$

Allen's PER semantics

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$\texttt{Definition}\ \mathrm{per\_function}\ (\mathit{ts} : \texttt{cts}) : \texttt{cts} := \ldots$

Closure

$\texttt{Inductive}\ \mathsf{close}\ (\mathit{ts} : \texttt{cts}) : \texttt{cts} := \ldots$

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Universes

$\texttt{Fixpoint}\ \mathrm{univi}\ (i : \mathsf{nat}) : \texttt{cts} := \ldots$

Allen's PER semantics

```
Fixpoint univi (i : nat) (T T' : CTerm) (eq : per) : Prop :=
  match i with
  | 0 ⇒ False
  | S n ⇒
    ...
    eq ⇐2⇒ (fun A A' ⇒ {eqa : per , close (univi n) A A' eqa})
    ...
end.
```

Has to be in Prop, otherwise we can only define a finite number of universes

Allen's PER semantics

`Definition` univ $T\ T'\ eq := \{i : \mathsf{nat}\ ,\ \mathrm{univi}\ i\ T\ T'\ eq\}$.

`Definition` nuprl := close univ.

$$t_1{\equiv}t_2{\in}T \ = \ \{eq : \mathrm{per}\ ,\ \mathrm{nuprl}\ T\ T\ eq \times eq\ t_1\ t_2\}$$

$$T{\equiv}T' \ = \ \{eq : \mathrm{per}\ ,\ \mathrm{nuprl}\ T\ T'\ eq\}$$

Interesting fact: $n{:}\mathbb{N} \to \mathbb{U}(n)$ is a Nuprl type

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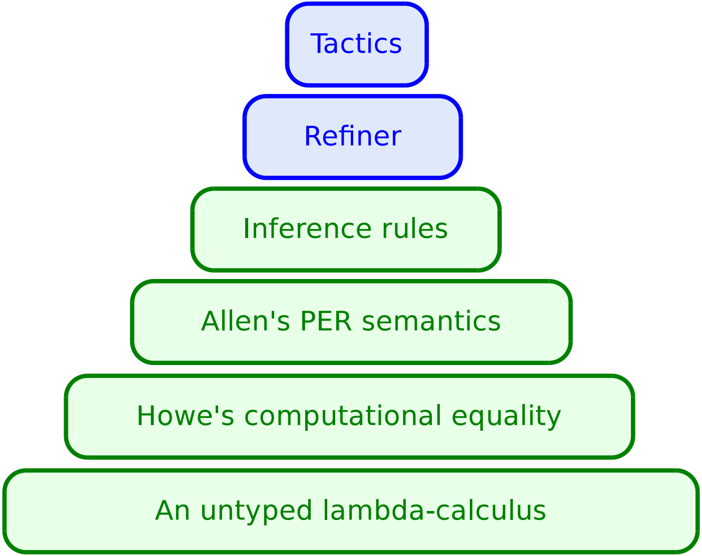

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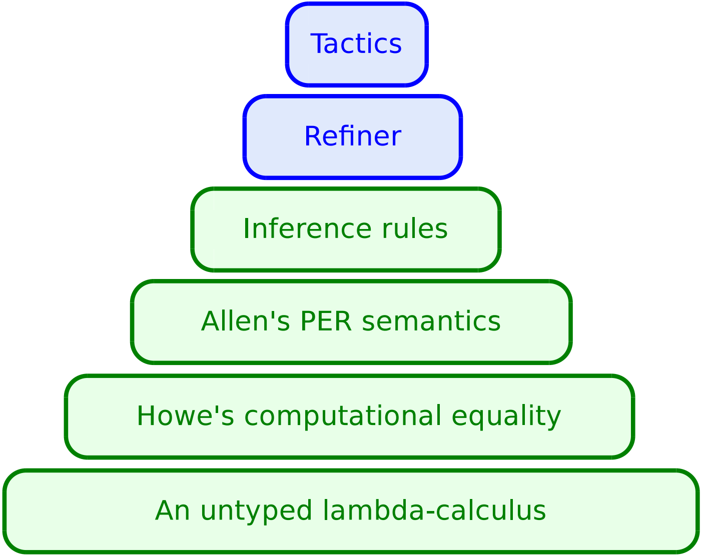

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Inference rules

We verified over 70 rules

The more rules the better

- ➲ Expose more of the metatheory
- ➲ Encode Mathematical knowledge

Gives us the basis to formally define a refiner

What next?

Adding new types

Extend our formalization with a library of definitions

Adding new computations

Build a verified refiner

Write a parser

Type checker/type inferencer?

Build a proof assistant

Implement Allen's semantics of Atoms

What about Mendler's recursive types?

What can you do with it?