Verified Efficient Implementation of Gabow's Strongly Connected Component Algorithm

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Motivation

- Verify algorithm that computes SCCs of a digraph
- Variants/Applications of algorithm
    - Enumerate SCCs
    - Emptiness check of Generalized Büchi-Automata
    - ...
- Re-use formalization between variants
- Generate efficiently executable code

Outline

1. Gabow's SCC Algorithm
2. Isabelle/HOL Formalization
3. Performance Evaluation

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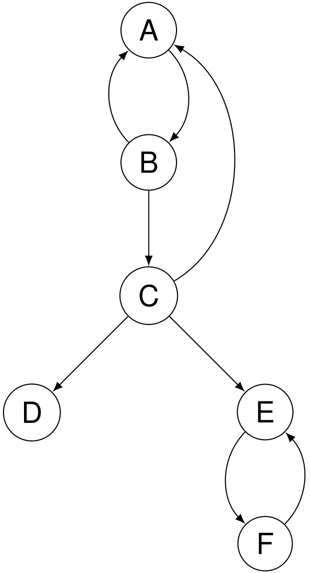
Strongly Connected Components

- SCC is maximal set of mutually reachable nodes



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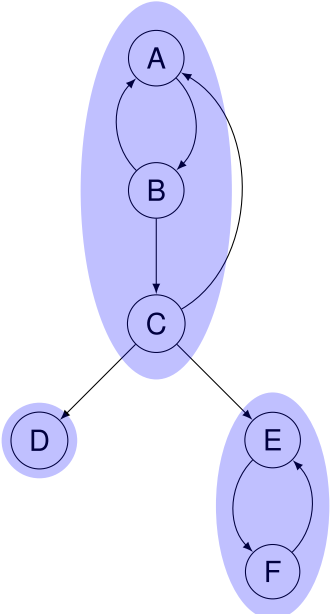
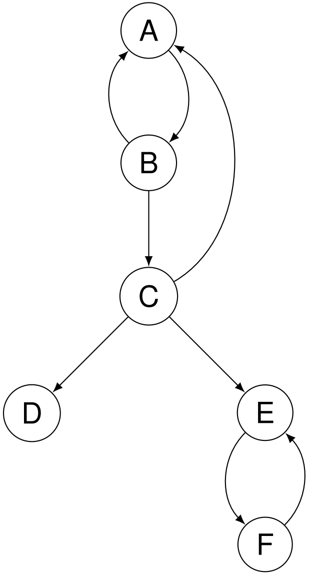


Path-Based Algorithms

- Depth first search
- On back edge, collapse nodes of induced cycle
- Eventually, each node represents SCC

Path-Based Algorithm Example



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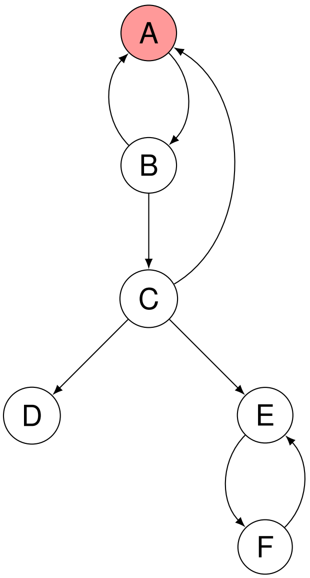

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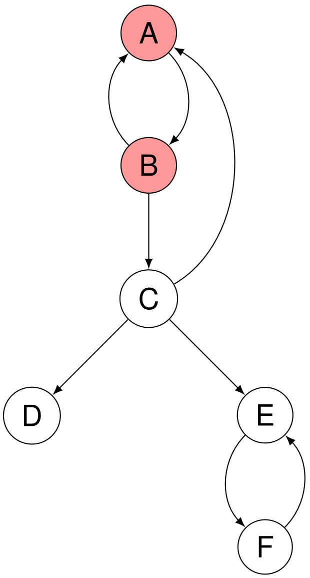

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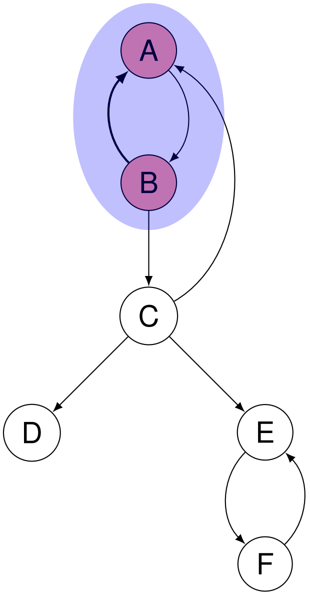

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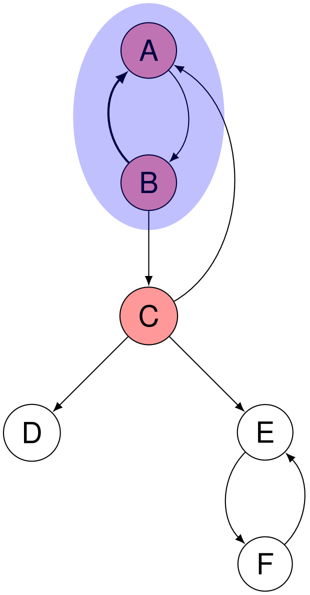

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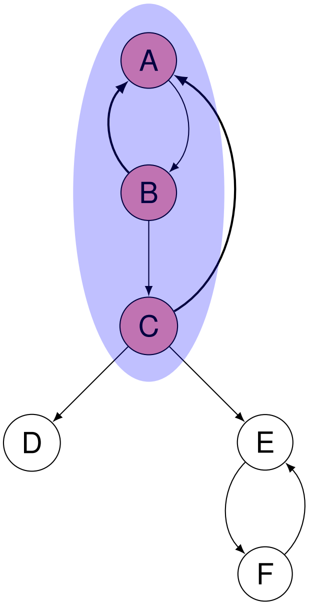

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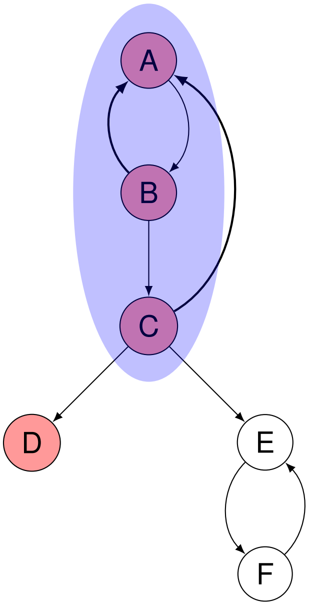

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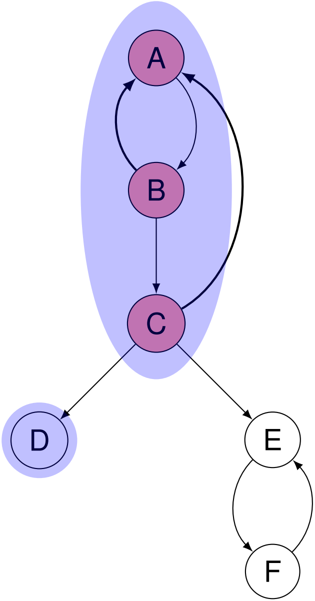

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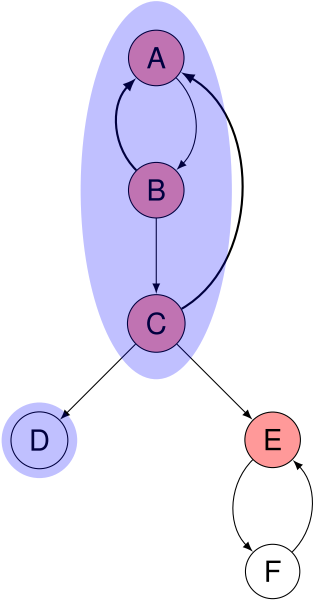

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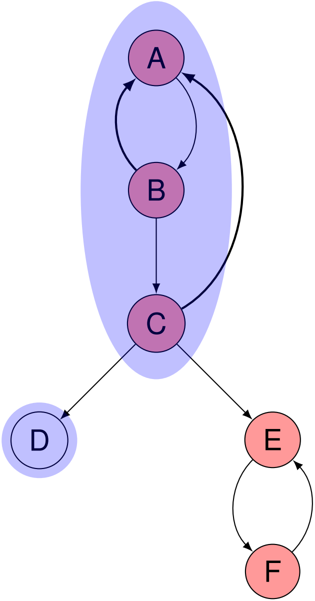

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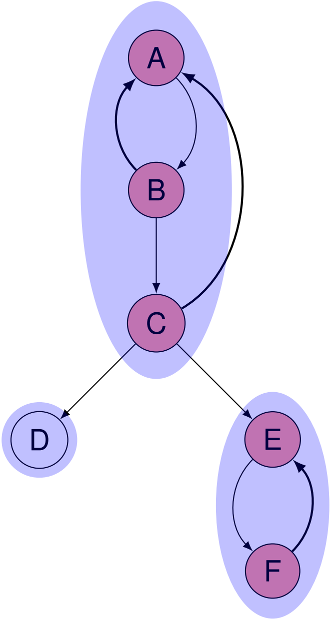

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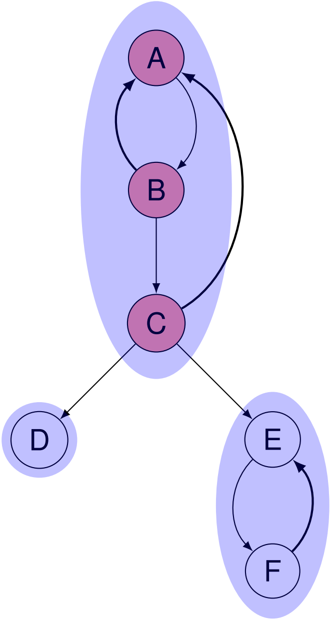

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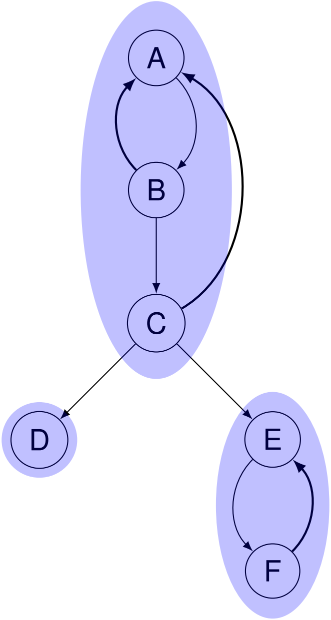
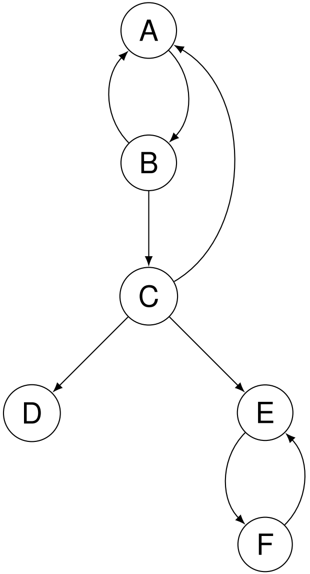


Gabow's Data Structure

- How to maintain collapsed nodes on stack?
- Use *boundary stack*
  - contains indexes of bounds between collapsed nodes
- Yields linear-time algorithm

Gabow's Data Structure Example



DFS stack:
Boundary stack:

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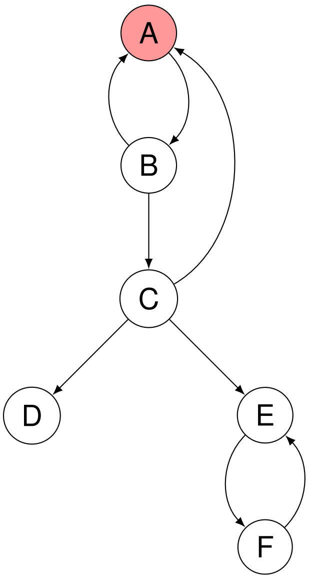

DFS stack: A
Boundary stack: 0

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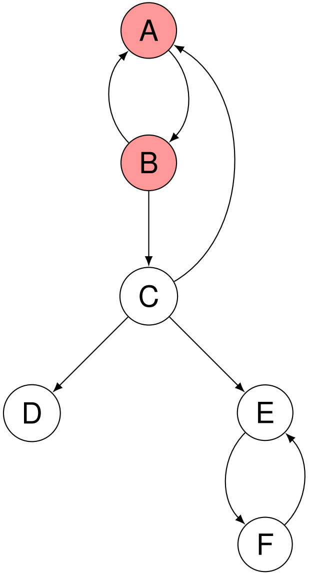

DFS stack:	A	B
Boundary stack:	0	1

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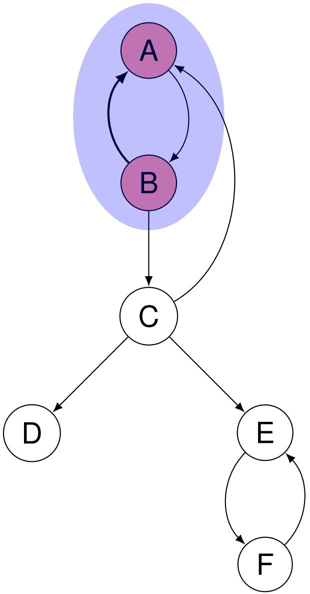

DFS stack: A B
Boundary stack: 0

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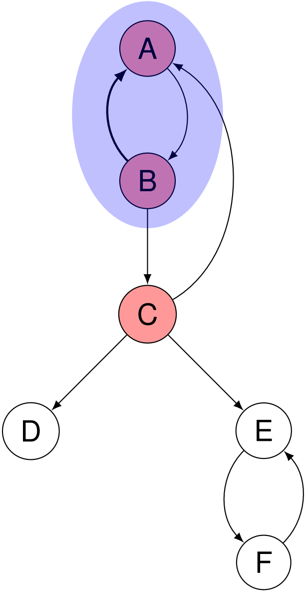

DFS stack:	A	B	C
Boundary stack:	0		2

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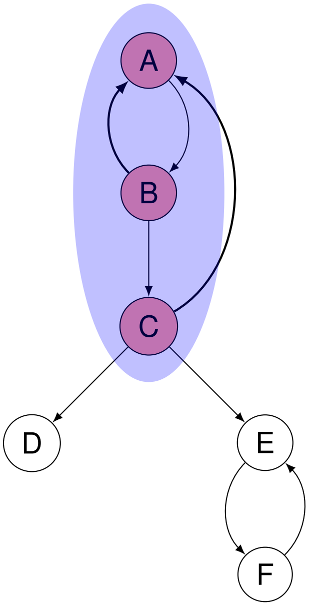

DFS stack: A B C
Boundary stack: 0

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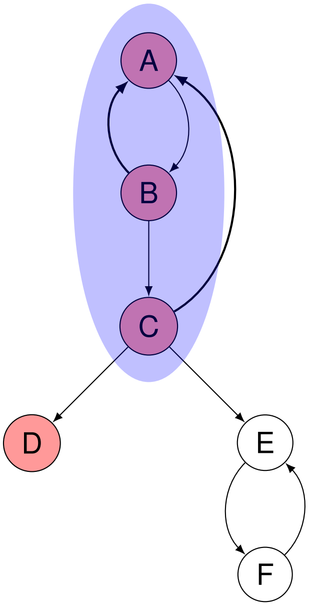

DFS stack:	A	B	C	D
Boundary stack:	0			4

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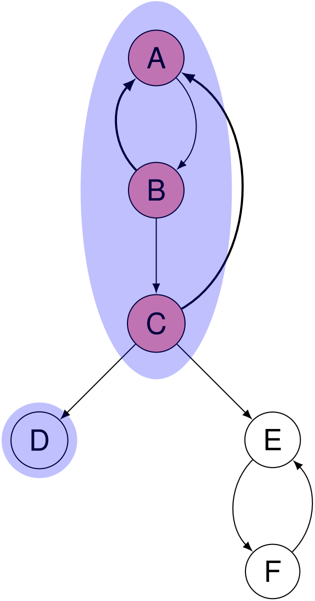

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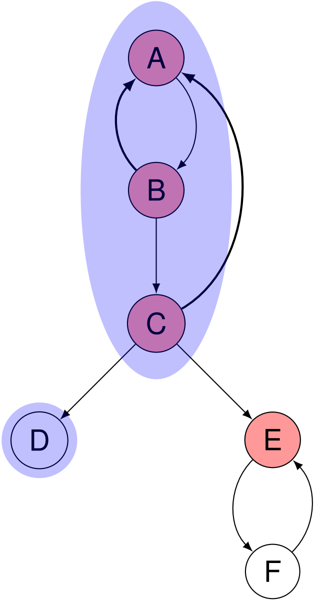

DFS stack:	A	B	C	E
Boundary stack:	0			4

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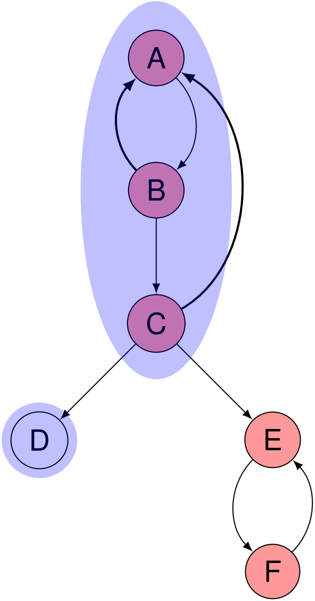

DFS stack:	A	B	C	E	F
Boundary stack:	0			4	5

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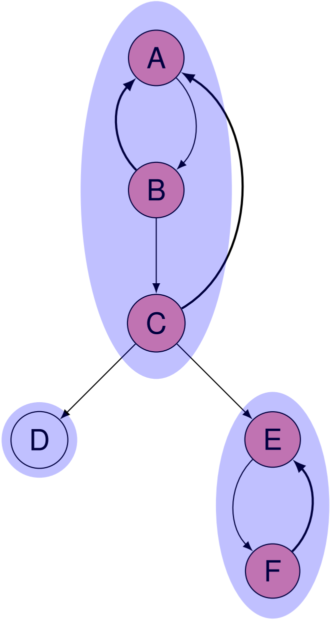

DFS stack:	A	B	C	E	F
Boundary stack:	0			4	

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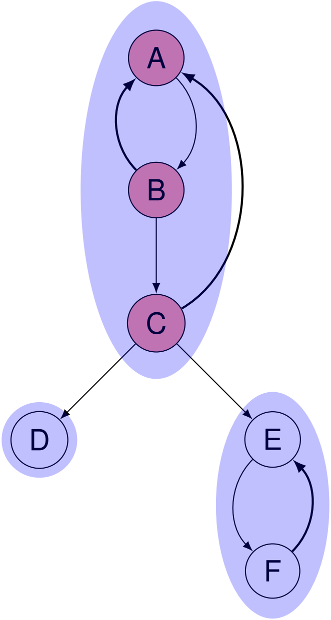

DFS stack: A B C
Boundary stack: 0

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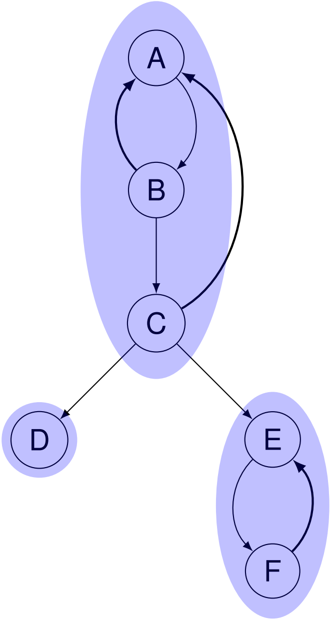

DFS stack:
Boundary stack:

Outline

Re-usable Formalization

- Goal: Formalize family of SCC-based algorithms
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- Reuse this formalization for actual algorithms
- Utilize existing Isabelle technologies
  - Collection Framework, Refinement Framework, Autoref tool
  - Code generator, locales

Design of the Formalization

Skeleton Specification

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$$\text{Skeleton Specification} \;\xrightarrow{\sqsubseteq}\; \text{Abstract Skeleton Algorithm}$$

Design of the Formalization

$$\begin{tikzcd}
\text{Skeleton Specification} \arrow[d, "\sqsubseteq"] \\
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\text{Gabow's Implementation}
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Isabelle Refinement Framework

- Nondeterministic monadic programs

```
skeleton ≡ do {
  let D = {};
  r ← FOREACHi outer_invar V0 (λv0 D0. do {
    if v0∉D0 then do {
      let s = initial v0 D0;
      (p,D,pE) ← WHILEIT (invar v0 D0) (λ(p,D,pE). p ≠ []) (λ(p,D,pE).
      do {
        (vo,(p,D,pE)) ← select_edge (p,D,pE);
        case vo of
          Some v ⇒
            if v ∈ ⋃set p then RETURN (collapse v (p,D,pE))
            else if v∉D then RETURN (push v (p,D,pE))
            else RETURN (p,D,pE)
        | None ⇒ RETURN (pop (p,D,pE))
      }) s;
      RETURN D
    } else RETURN D0
  }) D;
  RETURN r }
```

Isabelle Refinement Framework

- Nondeterministic monadic programs
- Supports stepwise refinement
- Verification Condition Generator

```
lemma "skeleton_impl ≤ ⇓oGS_rel skeleton"
  unfolding skeleton_impl_def skeleton_def
  by (refine_rcg skeleton_refines)
    (vc_solve (nopre) solve: asm_rl I_to_outer
      simp: skeleton_refine_simps)
```

Autoref-Tool and Collections Framework

- Automatic Refinement Tool (Autoref)
  - Parametricity-based approach to data refinement
  - Automatic synthesis of implementation from abstract program
- Isabelle Collection Framework
  - Efficient data structures (Array, Hash-Table, Bitvector, . . .)
  - Generic Algorithm Library
  - Integrated with Autoref

```
schematic_lemma skeleton_code_aux:
  "(RETURN ?skeleton_tr,skeleton_impl) ∈ ⟨oGSi_rel⟩nres_rel"
  unfolding ... by autoref
export_code skeleton_tr in SML file "gabow.sml"
```

Re-use of Invariants

- Exploit locale mechanism to define extended invariants
- Set up VCG: Only preservation of extension needs to be proved

```
locale invar -- "Invariants of Skeleton"
locale cscc_invar_ext -- "Additional invariants"
locale cscc_invar = invar + cscc_invar_ext -- "Combined invariant"

lemma cscc_invarI:
  assumes "invar s"
  assumes "invar s ⟹ cscc_invar_ext (l,s)"
  shows "cscc_invar (l,s)"
```

Re-use of Refinements

- Use basic operations in extended algorithm
- Re-use refinements for basic operations

```
compute_SCC ≡ …
| None ⇒ do {
    (* No more outgoing edges from current node on path *)
    ASSERT (pE ∩ last p × UNIV = {});
    let V = last p;
    let (p,D,pE) = pop (p,D,pE);
    let l = V#l;
    RETURN (l,p,D,pE)
  } …
lemma compute_SCC_impl_refine: "compute_SCC_impl ≤ ⇓Id compute_SCC"
proof -
  …
  show ?thesis
    unfolding compute_SCC_impl_def compute_SCC_def
    apply (refine_rcg … pop_refine …)
    by (vc_solve …)
qed
```

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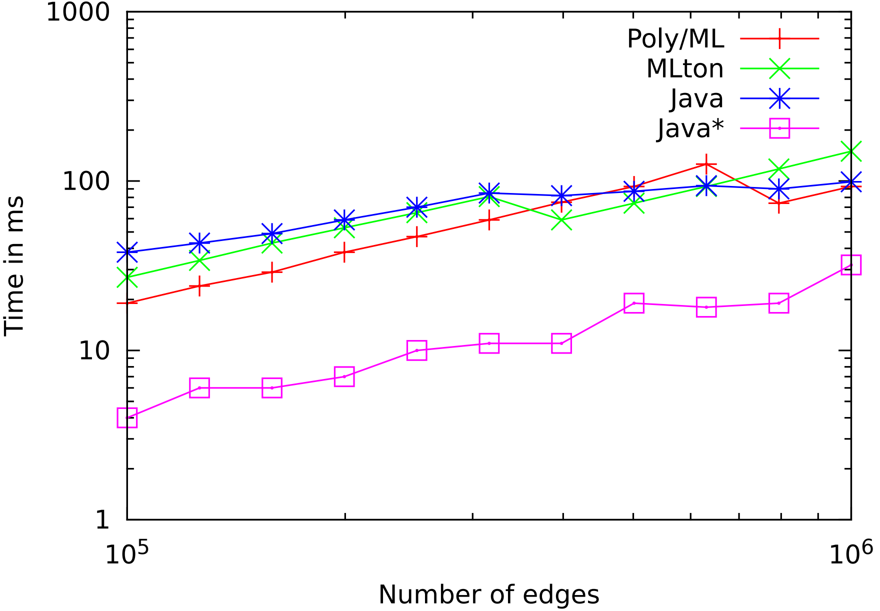
Benchmark against Java Reference Implementation



Conclusions

- Efficient, extensible formalization of Gabow's Algorithm
  - Performance comparable to Java implementation ($\times 3 \ldots \times 4$)
  - Variants: Enumerate SCCs, emptiness check for GBA
- Used by the CAVA fully verified LTL model checker [CAV '13]
- Example of verified algorithm design in Isabelle/HOL
  - Using Collection/Refinement/Autoref framework [ITP '10,'12,'13]
  - Refinement separates algorithmic ideas from implementation
  - Sharing of proofs between variants of the algorithm

Questions

Questions?
Remarks?