The Chain Rule and Friends in ACL2(r)

Rump Presentation

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The Main Result

We have verified in ACL2(r) the familiar algebraic differentiation rules:

- $(f+g)'(x) = f'(x) + g'(x)$.
- $(-f)'(x) = -f'(x)$.
- $(f \cdot g)'(x) = f(x)g'(x) + f'(x)g(x)$.
- $(1/f)'(x) = -f'(x)/(f(x) \cdot f(x))$.
- $(f \circ g)'(x) = f'(g(x))g'(x)$.
- $(f^{-1})'(x) = 1/f'(f^{-1}(x))$, where f^{-1} is the (compositional) inverse of f.

The Motivation

- These differentiation rules are useful in practice, for example when using Taylor's Theorem

The Real Motivation

- In other work, we showed how to introduce inverse functions in ACL2(r)
- We wanted to find the derivatives of inverse functions, e.g. $\ln'(x) = 1/x$
- It's obvious how to do it:

$$\begin{aligned}
(f \circ f^{-1}) &= id \\
(f \circ f^{-1})' &= id' \\
(f^{-1})' \cdot (f' \circ f^{-1}) &= 1 \\
(f^{-1})' &= \frac{1}{f' \circ f^{-1}}
\end{aligned}$$

Implementation Details

- All functions are introduced using **encapsulate**
- This means there are many almost identical **encapsulate**s, e.g., for f_1 and f_2 in $(f_1 + f_2)' = f_1' + f_2'$
- We designed the theorems so that they can be composed—this way, we can use the results to find the derivatives of complex functions
- In some cases, it is important that $f(x) \neq 0$ or even that $f'(x) \neq 0$. These hypotheses are added to the constraints of the function (over the intended domain)
- This preserves the compositionality of the theorems, since the conclusions exactly match the **encapsulate** constraints

The Big Disappointment

- After proving the chain rule, we set out to derive inverse functions
- It turns out that the story we presented earlier was flawed:

$$\begin{aligned}
(f \circ f^{-1}) &= id \\
(f \circ f^{-1})' &= id' \\
(f^{-1})' \cdot (f' \circ f^{-1}) &= 1 \\
(f^{-1})' &= \frac{1}{f' \circ f^{-1}}
\end{aligned}$$

- It shows what the derivative must be, but it does not guarantee that the derivative exists
- So we had to show that from first principles as well

Future Work, Part I

- We want to extend these results to the differentiation of complex functions
- This is necessary, since sine and cosine are defined in terms of e^{ix}
- And we want to reason about sine, cosine, and other curves in the e^{ix} family
- (See next year's workshop for details)

Future Work, Part II

- In principle, we can use these theorems to find the derivatives of very complex functions
- In practice, this is very tedious
- We would like to define a macro (**defderiv**?) that generates the derivative of a function and the associated proofs automatically