Area: Computer Theory

Three questions each from three parts

Answer five of the nine questions
Part 1. Foundations of Computing
Let $f$ be the function recursively defined on the nonnegative integers by

$$f(k, n) \overset{\text{def}}{=} \begin{cases} n + 1 & \text{if } k = 0 \\ f(k - 1, 1) & \text{else if } n = 0 \\ f(k - 1, f(k, n - 1)) & \text{else} \end{cases}$$

It is clear that for every nonnegative integer $n$, $f(0, n) = n + 1$. Use mathematical induction to prove for every nonnegative integer $n$, $f(1, n) = n + 2$. 
Show that if the following function for computing $n!$ halts, then it is correct. Clearly state and prove the conditional theorems required to show (partial) correctness.

function fac(n:nonneg int):positive int
  f:positive int
  i:positive int

  f := 1
  i := 2

  \{(f = (i-1)! \text{ and } i \leq n+1) \text{ or } (f = 1 \text{ and } i > n+1 \text{ and } n = 0)\}

  while i \leq n do
    f := f * i

    i := i + 1
  end while

  return(f)
Let $L$ be the set of all nonempty strings over \{a, b\} where the $a$'s and $b$'s alternate. Assume the strings $a$ and $b$ are in $L$,

- Give an inductive definition for $L$. You may use the string operations \texttt{append on the left, head, tail, and } \texttt{=} \texttt{(either to test for the empty string or to compare string elements). Be sure to apply head and tail only to nonempty strings.}

- Give a regular expression that matches exactly the elements of $L$. 
Part 2. Theory of Computation
Let $f$ be the function recursively defined on the nonnegative integers by

$$f(k, n) \overset{\text{def}}{=} \begin{cases} n + 1 & \text{if } k = 0 \\ f(k - 1, 1) & \text{if } n = 0 \\ f(k - 1, f(k, n - 1)) & \text{else} \end{cases}$$

Use the *Universality, Parameter, and Fixed Point Theorems* (see Davis, Sigal, and Weyuker, chapter 4) to prove there is a partially computable function $f$ satisfying the given recursive definition.
Let $F$ and $G$ be the functions recursively defined on the nonnegative integers by

$$
F(0) = 1 \\
G(0) = 1 \\
F(n + 1) = F(n) + G(n) \\
G(n + 1) = F(n) \cdot G(n)
$$

Prove $F$ and $G$ are primitive recursive functions.
Use Rice’s Theorem (see Davis, Sigal, and Weyuker, chapter 4) to show none of the following sets is recursive.

\[ A = \{ n \in N \mid (\forall x \in N) [\Phi(x, n) \uparrow] \} \]
\[ B = \{ n \in N \mid (\exists x \in N) [\Phi(x, n) > x^2] \} \]
\[ C = \{ n \in N \mid \Phi(x, n) \text{ is defined for all but finitely many } x \} \]

Here \( N \) is the set of all nonnegative integers.