Verified Decision Procedures for Equivalence of Regular Expressions

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Background

Recent series of papers presenting such decision procedures verified in Coq, Isabelle or Matita:

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They all look different but related . . .

This talk

- Unified framework

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- Derivation of all previous procedures as instances

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- Derivation of all previous procedures as instances
- Verification in Isabelle

Regular expressions

$$\textbf{datatype}\ \alpha\ rexp = \begin{array}{l} 0 \mid \\ 1 \mid \\ Atom\ \alpha \mid \\ \alpha\ rexp + \alpha\ rexp \mid \\ \alpha\ rexp \cdot \alpha\ rexp \mid \\ \alpha\ rexp^{*} \end{array}$$

Regular expressions

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\textbf{datatype}\ \alpha\ rexp = \begin{array}{l} 0 \mid \\ 1 \mid \\ Atom\ \alpha \mid \\ \alpha\ rexp + \alpha\ rexp \mid \\ \alpha\ rexp \cdot \alpha\ rexp \mid \\ \alpha\ rexp\ ^{*} \end{array}
$$

Semantics: $L :: \alpha\ rexp \to \alpha\ lang$
where $\alpha\ lang = \alpha\ list\ set$

How to prove $r \equiv s$

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1. Translate to DFAs A and B
2. Compare A and B
   - Standard algorithm:
     Minimize A and B, check isomorphism.
   - Easy alternative:
     Check for all reachable states (p, q) of $A \times B$ that p is final iff q is final.

Framework parameters

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Init	$init :: \alpha\ rexp \rightarrow \sigma$

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Assumptions:

$$\mathcal{L}(init(r)) = L(r)$$

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$$fin(s) \Leftrightarrow [] \in \mathcal{L}(s)$$

Equivalence checker

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\begin{aligned}
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$eqv\ r\ s \Longrightarrow L(r) = L(s)$

If the set of reachable states is finite:

Theorem

$L(r) = L(s) \Longrightarrow eqv\ r\ s$

Derivatives (Brzozowski 1964)

$$d :: \alpha \to \alpha\ rexp \to \alpha\ rexp$$

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- $d\ x\ r =$ "what is left after x has been read"
- Example: $d\ a\ (Atom(a) \cdot r) = 1 \cdot r$
- Semantics is left-quotient:
  $L(d\ x\ r) = \{w \mid xw \in L(r)\}$

$$d\ x\ 0 \;=\; 0$$

$$
\begin{aligned}
d\,x\,0 &= 0 \\
d\,x\,1 &= 0
\end{aligned}
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d\;x\;0 &= 0 \\
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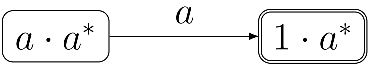
&\quad\; \textsf{else } d\;x\;r \cdot s \\
d\;x\;(r^*) &= d\;x\;r \cdot r^*
\end{aligned}
$$

Regular Expression $\rightsquigarrow$ DFA

$a \cdot a^*$

Regular Expression $\rightsquigarrow$ DFA



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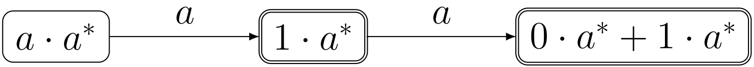

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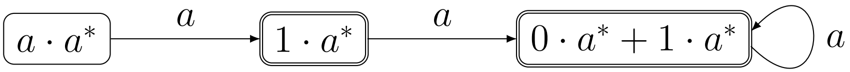

Finiteness

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Theorem (Brzozowski 1964)
The set $\{fold\ d\ w\ r \mid w \in \Sigma^*\}/{\equiv_{ACI}}$ is finite.

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How large?

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The set $\{fold\ d\ w\ r \mid w \in \Sigma^*\}/{\equiv_{ACI}}$ is finite.

How large? Brzozowski's proof yields $O(2^{\cdot^{\cdot^{\cdot^{2^n}}}})$

Instantiation of framework

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$$\sigma \ = \ \alpha \, rexp$$

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$$\begin{aligned} \sigma &= \alpha\, rexp \\ init(r) &= r \end{aligned}$$

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Finiteness:

- Not immediate from Brzozowski's theorem

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Finiteness:

- Not immediate from Brzozowski's theorem
- Open for stronger normalization functions

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where $\{r_1, \ldots, r_n\} \odot s = \{r_1 \cdot s, \ldots, r_n \cdot s\}$

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Starting from a regular expression r

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$\Longrightarrow 2^{|r|_{at}+1}$ sets of regular expressions reachable

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Functional implementation by
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Functional implementation by
Fischer, Huch & Wilke [ICFP 2009]:

- Replace sets of positions by marked regular expressions: $Atom(bool, \alpha)$
- Only matching, not $\equiv$, no proofs

Example: $(a \cdot a + a \cdot b)^*$

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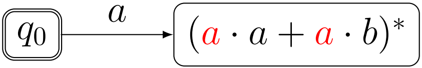

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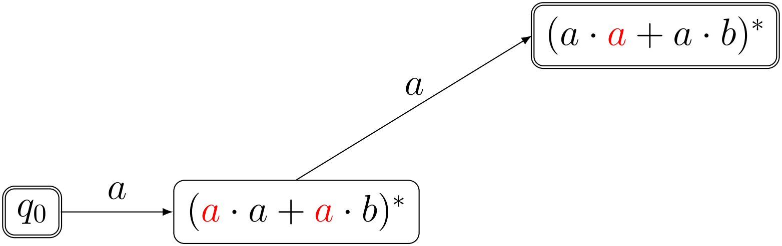

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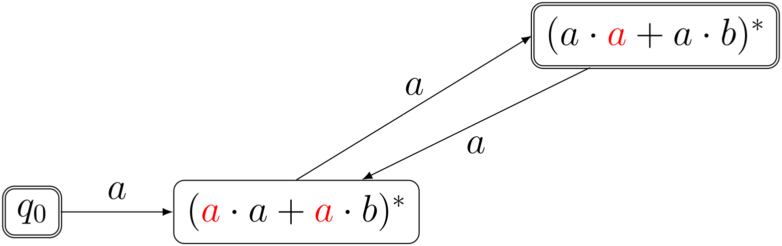

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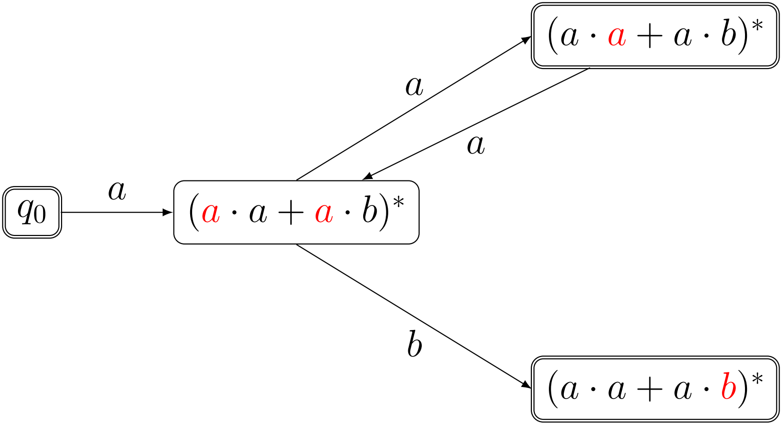

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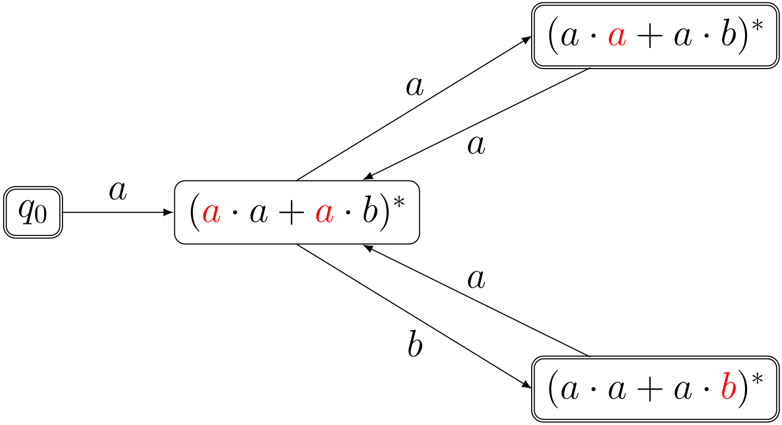

Instantiation of framework

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$$\sigma \;=\; bool \times (bool \times \alpha)\, rexp$$

Instantiation of framework

$$\begin{aligned} \sigma &= bool \times (bool \times \alpha)\, rexp \\ init(r) &= (True,\ map\,(\lambda a.\,(False, a))\, r) \end{aligned}$$

Instantiation of framework

$$\begin{aligned} \sigma &= \mathit{bool} \times (\mathit{bool} \times \alpha)\ \mathit{rexp} \\ \mathit{init}(r) &= (\mathit{True},\ \mathit{map}\ (\lambda a.\ (\mathit{False}, a))\ r) \\ \delta\ x\ (m, r) &= (\mathit{False},\ \mathit{read}\ x\ (\mathit{follow}\ m\ r)) \end{aligned}$$

Instantiation of framework

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\delta\ x\ (m, r) &= (\mathit{False},\ \mathit{read}\ x\ (\mathit{follow}\ m\ r)) \\
\mathit{fin}(m, r) &= \ldots \\
\mathcal{L}(m, r) &= \ldots
\end{aligned}
$$

Conceptually,
the marks in McNaugton/Glushkov/Fisher
are *after* the atoms

Marked regular expressions II

Asperti [ITP 2012]:

- Verified $\equiv$-checker via marked $rexp$ in Matita

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Asperti [ITP 2012]:

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- Says he has formalised McNaughton & Yamada
- ...but he invented his own variation:
  Puts the mark *before* the atom

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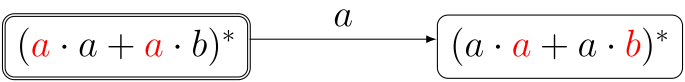

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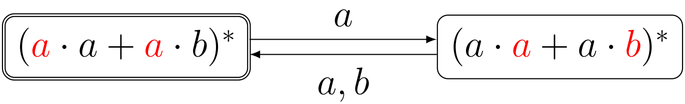

Instantiation of framework

Similar but a bit more complicated

Before vs After

Before vs After

Transitions can be decomposed into two steps:

- Before: $read; follow$

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Theorem

The before-automaton is a homorphic image of the after-automaton.

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Theorem

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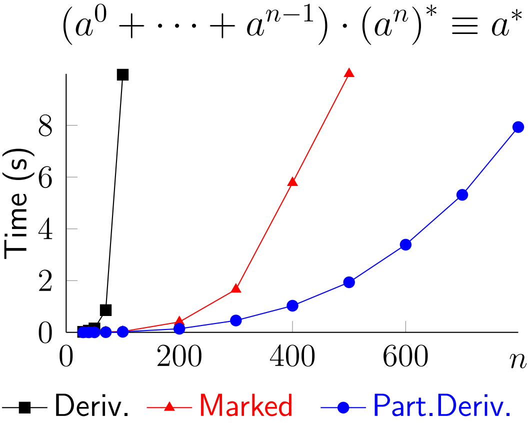
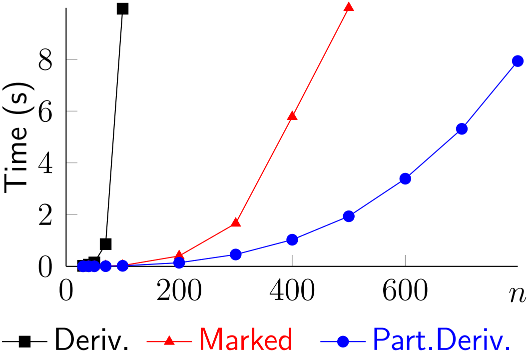
Proof idea due to Helmut Seidl.


$(a^0 + \cdots + a^{n-1}) \cdot (a^n)^* \equiv a^*$
Time (s)
0
2
4
6
8
0
200
400
600
n
Deriv.
Marked
Part.Deriv.

$$(a^0 + \cdots + a^{n-1}) \cdot (a^n)^* \equiv a^*$$



For randomly generated examples:

Deriv. $\gg$ Part.Deriv. $\gg$ Fischer, Asperti

Extended regular expressions

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Complement and intersection:

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- Trivial for derivatives (Brzozowski)

Extended regular expressions

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- Harder for partial derivatives (Champarnaud and Mignot)

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. . . and projection:

Extended regular expressions

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  $\leadsto$ decision procedure for MSO on finite strings

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